Lecture
Computational Intelligence
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Plans for Today

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Subsets

We already defined
\[ A \subseteq B \iff \forall x: A(x) \leq B(x). \]

Note: This is crisp!

Is it plausible?

Example:
\[ \mathcal{U} = \mathbb{N}_0 \]
\[ A(x) := \begin{cases} \frac{1}{x^2} & \text{if } x > 0 \\ a & \text{if } x = 0 \end{cases} \quad B(x) := \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ b & \text{if } x = 0 \end{cases} \quad a > b \]

Clearly, \( \forall x \in \mathcal{U} \setminus \{0\}: A(x) \leq B(x). \)

But because \( a > b, \) \( \neg (A \subseteq B). \)
Fuzzy Subsets

A different perspective:

Crisp sets: \( A \subseteq B \iff \forall x: x \in A \implies x \in B \)

Definition:
For any lower semicontinuous \( t \)-norm \( t \):
\[
A \subseteq_t B :\iff \forall x \in U: \phi_t(A(x), B(x)) = \inf\{\phi_t(A(x), B(x)) \mid x \in U\}
\]
\[
A \equiv_t B :\iff t(A \subseteq_t B, B \subseteq_t A)
\]

What does \( \forall \) mean for fuzzy sets?

Definition:
\[
\forall x: F(x) := \inf\{F(u) \mid u \in U\}
\]
\[
\exists x: F(x) := \sup\{F(u) \mid u \in U\}
\]
Subsets — Examples

\[ A(x) := \begin{cases} \frac{1}{x^2} & \text{if } x > 0 \\ a & \text{otherwise} \end{cases} \quad B(x) := \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ b & \text{otherwise} \end{cases} \quad a > b \]

\[ A \subseteq_t B = \inf \{ \phi_t(A(x), B(x)) \mid x \in \mathcal{U} \} = \inf \{ \sup \{ w \mid t(A(x), w) \leq B(x) \} \mid x \in \mathcal{U} \} \]

\[ A \subseteq_{t_m} B = \inf \{ \sup \{ w \mid \min\{A(x), w\} \leq B(x) \} \mid x \in \mathcal{U} \} = b \]

\[ A \subseteq_{t_l} B = \inf \{ \sup \{ w \mid \max\{0, A(x) + w - 1\} \leq B(x) \} \mid x \in \mathcal{U} \} = b + (1 - a) \]

\[ A \subseteq_{t_p} B = \inf \{ \sup \{ w \mid A(x)w \leq B(x) \} \mid x \in \mathcal{U} \} = \frac{b}{a} \]

\[ A \subseteq_{t_d} B \]

\[ = \inf \{ \sup \{ w \mid [\max\{A(x), w\} = 1] \cdot \min\{A(x), w\} \leq B(x) \} \mid x \in \mathcal{U} \} \]

\[ = \begin{cases} b & \text{if } a = 1 \\ 1 & \text{otherwise} \end{cases} \]
Relations

Crisp relation $R$ over $U$ is a function $R : U \times U \rightarrow \{0, 1\}$.

Fuzzy relation $R$ over $U$ is a function $R : U \times U \rightarrow [0, 1]$, i.e. fuzzy set over $U \times U$.

**Definition:** Let $R$ and $S$ be fuzzy relations and $t$ a $t$-norm:

- **domain** $\text{dom} (R) := \{x \mid \exists y : R(x, y) > 0\}$
- **range** $\text{rg} (R) := \{y \mid \exists x : R(x, y) > 0\}$
- **inverse** $R^{-1} : U \times U \rightarrow [0, 1]$ defined by $R^{-1}(y, x) := R(x, y)$
- **product** $R \circ_t S := \{(x, y) \mid \exists z : t(R(x, z), S(z, y))\}$

**Note:** All these sets are fuzzy!
Properties of Fuzzy Relations

**Definition:** Let \( t \) be some lower semicontinuous \( t \)-norm, \( n \) some negation.

Fuzzy relation \( R \) is called (with respect to \( t \) and \( n \))

1. **reflexive** iff \( \forall x : R(x, x) = 1 \)
2. **\( n \)-irreflexive** iff \( \forall x : n(R(x, x)) = 1 \)
3. **\( t \)-transitive** iff \( \forall x, y, z : \phi_t(t(R(x, y), R(y, z)), R(x, z)) = 1 \)
4. **symmetric** iff \( \forall x, y : \phi_t(R(x, y), R(y, x)) = 1 \)
5. **\( t \)-antisymmetric** iff
   \[ \forall x, y : \phi_t(t(R(x, y), R(y, x)), [x = y]) = 1 \]
6. **\( t \)-asymmetric** iff \( \forall x, y : n(t(R(x, y), R(y, x))) = 1 \)

**Note:** All these properties are crisp!

**Note:**
Removing ‘\( [\cdot] = 1 \)’ introduces fuzzy versions of the properties.
On Properties of Fuzzy Relations

Why does it say “symmetric” and not “$t$-symmetric?”

$[\forall x, y : \phi_t(R(x, y), R(y, x))] = 1$

$\iff \inf\{\phi_t(R(x, y), R(y, x)) \mid x, y \in U\} = 1.$

$(\phi_t(R(x, y), R(y, x)) = 1) \iff (R(x, y) \leq R(y, x))$

for any $t$-norm $t$

Thus, “symmetric” is independent of the choice of the $t$-norm.

Note: This is not true for the fuzzy version of “symmetric.”
Properties of $t$-norms Reconsidered

\[ \Delta_U := \{ (x, x) \in U \times U \} \]

**Theorem:** ($R$ fuzzy relation, $t$ lower semicontinuous $t$-norm, $n$ negation)

1. $R$ reflexive iff $\Delta_U \subseteq R$
2. $R$ $n$-irreflexive iff $\left( R \cap_t \Delta_U \right)^{C_n} \equiv \emptyset$
3. $R$ $t$-transitive iff $R \circ_t R \subseteq R$
4. $R$ symmetric iff $R \subseteq R^{-1}$
5. $R$ $t$-antisymmetric iff $R \cap_t R^{-1} \subseteq \Delta_U$
6. $R$ $t$-asymmetric iff $\left( R \cap_t R^{-1} \right)^{C_n} \equiv 1$
Motivation

We have knowledge about

- fuzzy logic,
- fuzzy sets,
- fuzzy numbers, and
- fuzzy relations.

We want to model and control technical systems.

We begin with a methodology for modeling.
Linguistic Variables

**Definition:** A linguistic variable $V$ is characterized by

- its name $x$,
- a universe $\mathcal{U}$,
- a term set $T(x)$,
- a syntactic rule $G$ for generating names of values of $x$, and
- a semantic rule $M$ for associating meanings with values.

**Example:** Describe the speed of a car.

- name $x = $ speed
- universe $\mathcal{U} = [0; 250]$ (possible crisp values)
- term set $T(x) = \{\text{slow, moderate, fast, too fast}\}$
- syntactic rule $G$: a fuzzy set for each term from $T(x)$
- semantic rule $M$: an interpretation for each term from $T(x)$
Linguistic Variable Speed

- slow
- moderate
- fast
- too fast
Standard Term Sets

Linguistic variables are a very general tool.

It helps to have standard solutions for standard situations.

\[ T(x) = \{ \text{negative big (NB), negative medium (NM), negative small (NS), zero (ZE), positive small (PS), positive medium (PM), positive big (PB)} \} \]
Linguistic Hedges

In natural language, properties can be modified using words like slightly, fairly, very . . .

We can adopt this in fuzzy modeling using modifiers, also called linguistic hedges.

A linguistic hedge $\text{MOD}$ is a function mapping fuzzy sets to fuzzy sets, i.e. $\text{MOD} : \mathcal{F} \rightarrow \mathcal{F}$ ($\mathcal{F}$ set of all fuzzs sets over universe $\mathcal{U}$).

Possible properties of linguistic hedges $\text{MOD}$:

- expanding: $\forall F : F \subseteq \text{MOD}(F)$
- compressing: $\forall F : \text{MOD}(F) \subseteq F$
- closed: $\forall F : \text{MOD}(\text{MOD}(F)) = \text{MOD}(F)$

Often: $\text{MOD}(F)(x) := (\text{MOD}(F(x)))$, where $\text{MOD} : [0, 1] \rightarrow [0, 1]$. 
Linguistic Hedge “VERY”

Typical definition: \( \text{VERY}(u) = u^2 \)
Linguistic Hedge “MORE OR LESS”

Typical definition: \( \text{MORE OR LESS}(u) = \sqrt{u} \)
Fuzzy IF-THEN-Rules

Assume technical system described by means of linguistic variables.

How can we define control of this system?

**Example:** crash prevention

- **input:** current speed, current distance to obstacle
- **output:** force used for braking

Describe control as fuzzy IF-THEN-rule:
IF speed=high and distance=small THEN brake sharply.

**Clearly:** We can implement “and” using some $t$-norm.
A Fuzzy IF-THEN-Rule

IF $X = A$ THEN $Y = B$

$X, Y$: linguistic variables
$A, B$: linguistic terms

Fuzzy IF-THEN-Rule constitutes a relation.

Possible interpretation:
Specific input defines the activation degree of the rule.

In general, input may be any fuzzy set $M$.

$$\text{act}(M, A) = \sup_{u \in \mathcal{U}} \{ t(M(u), A(u)) \}$$

If $M$ is singleton (only 1 for exactly one $x$), $(\text{act})(M, A) = A(x)$. 
A Fuzzy IF-THEN-Rule

**IF X = A THEN Y = B**

\(X, Y\): linguistic variables (over universes \(\mathcal{U}, \mathcal{U}'\))

\(A, B\): linguistic terms

We have fuzzy sets for ‘\(A\)’ and ‘\(B\)’ (over \(\mathcal{U}\) resp. \(\mathcal{U}'\)).

We need a fuzzy set \(R\) over \(\mathcal{U} \times \mathcal{U}'\) for the relation

**IF X = A THEN Y = B.**

Possible definitions:

- Read IF-THEN as implication \(\rightarrow\) \(R(x, x') := \phi_t(A(x), B(x'))\)
- Read IF-THEN as \(t\)-norm \(\rightarrow\) \(R(x, x') := t(A(x), B(x'))\)

Justification:

Read ‘IF \(x = a\) THEN \(y = b\)’ as

It is true that \(x = a\) holds and \(y = b\) holds.

**Result** a fuzzy set \(R(x') = R(act(M, A), x')\) for IF-THEN-rule

(depending on the input \(M\)) that serves as a result.
Reconsidering the Approach

Obvious Problems:

1. The input parameters will be crisp.
2. One fuzzy IF-THEN-rule will not be sufficient.
   \( \sim \) How do we integrate many IF-THEN-rules?
3. Finally, a crisp value for the braking force is needed.

Answers:

1. fuzzification
2. either (1) do inference for each rule and (2) aggregate later or (1) aggregate all rules and (2) do inference
3. defuzzification
Fuzzification

In general, we may convert crisp inputs into fuzzy sets arbitrarily.

Often, inputs are numbers $\leadsto$ fuzzy numbers

Most often, crisp inputs are taken as singletons.

Given such a crisp value, for each linguistic term $\in T(x)$ its membership function yields a degree of membership.

Thus, we get a vector of length $|T(x)|$ for one crisp value.
Fuzzification

Example: linguistic variable: speed
crisp input: 160 (in km/h)

Using 160 as a singleton yields:

Graph showing the membership grades for 'slow', 'moderate', 'fast', and 'too fast' with the crisp input 160.
Fuzzy Inference for a Single Rule

Example: IF speed=fast THEN brake=sharply.

crisp input: 160 (in km/h)

We already know: [fast(160)] = .54

Definition: sharply(x) = \[
\begin{cases}
0 & \text{if } x < 50 \\
1 & \text{if } x > 90 \\
\frac{x-50}{40} & \text{otherwise}
\end{cases}
\]

What do we get if we use

- \(\phi_{tm}\) for IF-THEN?
- \(\phi_{tp}\) for IF-THEN?
- \(t_m\) for IF-THEN?
Example continued

Example: IF speed = fast THEN brake = sharply.

crisp input: 160 (in km/h), \([\text{fast}(160)] = 0.54\)
Fuzzy Inference for Many Rules

**FIT A: First Inference, Then Aggregate.**

Do inference for each rule → one fuzzy set for each rule
Aggregate results. — often: $t$-conorm

**FATI: First Aggregate, Then Inference.**

Aggregate all IF-THEN-Rules.
Do inference for the aggregated rule.

Depending on operators, FATI and FITA may be equivalent.
Defuzzification

Our result is a fuzzy set.

In most applications, we need a crisp value.

For fuzzy set $M$ define crisp value $m$ by

- **center of gravity:** $m := \frac{\int \limits_{\mathcal{U}} u M(u) \, du}{\int \limits_{\mathcal{U}} M(u) \, du}$ for continuous $\mathcal{U}$

  $$m := \frac{\sum \limits_{\mathcal{U}} u M(u)}{\sum \limits_{\mathcal{U}} M(u)}$$ for discrete $\mathcal{U}$

- **first of maxima:** $m := \min \{ u \mid M(u) = \max \limits_{v} M(v) \}$

- **middle of maxima:** Let $\mathcal{N}$ be the set of maxima of $M$.

  $$m := \frac{\int \limits_{\mathcal{N}} u \, du}{\int \limits_{\mathcal{N}} 1 \, du}$$ for continuous $\mathcal{N}$

  $$m := \frac{1}{\mathcal{N}} \sum \limits_{u \in \mathcal{N}} u$$ for discrete $\mathcal{N}$
Defuzzification of Example

**Example:** IF speed$\rightarrow$fast THEN brake sharply.

crisp input: 160 (in km/h), [fast(160)] $= .54$

Center of Gravity: $82.16256$
General Fuzzy Controller

- Crisp input $x \in \mathcal{U}$
- Fuzzifier
  - Fuzzy set in $\mathcal{U}$
  - Fuzzy Rule Base
  - Fuzzy Inference Engine
  - Fuzzy set in $\mathcal{V}$
- Defuzzifier
  - Crisp output $y \in \mathcal{V}$
Mamdani Fuzzy Controller

Different types of fuzzy controllers are known.

Probably the best known: Mamdani (1974)

- use $t_m$ for ‘and’ in IF-part
- use FITA
- use $t_m$ for IF-THEN inference
- use $s_{tm}$ for aggregation
- use center of gravity defuzzification

Properties:

- intuitively understandable
- easy to construct
- does not support data-driven automatic construction
Design of Mamdani Controllers

**Heuristics for the construction Mamdani Controllers:**

- **Completeness:** For any possible input, have at least one rule.
- **Consistency:** Don’t have two rules with equal IF-part and different THEN-parts.
- **Computational complexity:** Use membership functions that allow for easy calculations of membership values. Particularly good: triangular and trapezoidal membership functions
- **Note:** Center of gravity defuzzification implies that extreme values cannot be obtained as output.
Implications for Inference

What changes if we use $\phi_t$ for IF-THEN?

Smaller values in IF-part yield larger results.

**New interpretation:**
Result is description of possibilities:
anything that cannot be ruled out.

Less useful in technical applications.
Takagi-Sugeno-Kang Fuzzy Controller

Takagi/Sugeno/Kang (1975):

Motivation:

- allow for automatic data-driven tuning of parameters
- reduce number of fuzzy rules needed

Change format of IF-THEN-rules:

- leave IF-part unchanged
- replace linguistic variable in THEN-part by linear function $f$

Consequence: Each rule describes conditions for application of specific linear model.

Aggregation yields aggregation of (independent) linear models.

No defuzzification necessary.
Takagi-Sugeno-Kang Fuzzy Controllers

Takagi-Sugeno-Kang fuzzy controllers have many parameters.

How can we find appropriate linear functions?

- use analytical models if available
- observe existing controller and derive linear model
- optimize numerically (for example using evolutionary algorithms)
- use some ‘learning approach’ (for example artificial neural nets)
Conclusion

- Fuzzy sets allow to formalize linguistic concepts (small, big, cold, warm, slow, fast, ...)
- Many possible operators on fuzzy sets: \( t \)-norm for conjunction, \( t \)-conorm for disjunction, \( \Phi \)-operator for implication
- Axiomatic approach allows to introduce new operators easily
- Fuzzy IF-THEN-rules formalize intuitive knowledge about control rules
- Fuzzy control allows to build controllers based on intuitive knowledge only