Lecture

Computational Intelligence

Winter Term 2007/2008

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28.11.2007 (Wednesday)
Last time we had...

- motivation “fuzziness”
- introduction to fuzzy sets
- fuzzy logic conjunction via $t$-norms
- fuzzy logic concrete $t$-norms
  Łukasiewicz, minimum, product, drastic, and others
- fuzzy logic disjunction via $t$-conorms
- introduction to fuzzy numbers
- fuzzy logic implication via $\phi$-operators
Plans for Today

1. Introduction
   Reminder

2. Fuzzy Logic
   Φ-operator
   Concrete Implications
   Negations

3. Fuzzy Control
   Introduction
   Fuzzy Sets
   Subset Relationship
Fuzzy Implications

Like disjunction and conjunction, we can introduce implication in an axiomatic way:

**Definition:** Let $t$ be some $t$-norm. An operator $p: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a **Φ-operator connected to $t$** iff for all $u, v, w \in [0, 1]$

1. $(v \leq w) \Rightarrow (p(u, v) \leq p(u, w))$
2. $t(u, p(u, v)) \leq v$
3. $v \leq p(u, t(u, v))$

hold.

**Easy to see:** These properties are fulfilled for Boolean $\land$ and $\Rightarrow$
Connecting $t$-norms and Φ-operators

**Theorem:** If $p$ is a Φ-operator connected to the $t$-norm $t$, then $p(u, v) = \sup\{w \mid t(u, w) \leq v\}$ holds for all $u, v \in [0, 1]$.

**Proof:** $t(u, p(u, v)) \leq v$ by definition. Thus, $p(u, v) \leq \sup\{w \mid t(u, w) \leq v\}$.

Assume $p(u, v) < \sup\{w \mid t(u, w) \leq v\}$.

Then, $\exists w_0: p(u, v) < w_0$ and $t(u, w_0) \leq v$.

Consider $p(u, t(u, w_0)) \leq p(u, v) < w_0$.

We have $w_0 \leq p(u, t(u, w_0))$ by definition.

Together: $w_0 \leq p(u, t(u, w_0)) \leq p(u, v) < w_0 \rightarrow$ contradiction

Thus, $p$ is uniquely connected to one $t$-norm.
t-norms and Φ-operators

Do we get a Φ-operator for any t-norm?

The notion of t-norms is very general. We restrict it with continuity in mind.

Definition: A t-norm t is called left continuous iff for all $u, v \in [0, 1]$ and all convergent sequences $(u_i)_{i \geq 1} \in [0, 1]$ with $u_i < u$ and $\lim_{i \to \infty} u_i = u$ we have:

$$\lim_{i \to \infty} t(u_i, v) = t(u, v).$$

Definition: A t-norm t is called lower semicontinuous iff for all $u, v \in [0, 1]$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u' \in (u - \delta, u]$:

$$t(u', v) > t(u, v) - \varepsilon.$$
Left Continuity and Lower Semicontinuity

**Theorem:** A $t$-norm is left continuous iff it is lower semicontinuous.

**Proof:**

“lower semicontinuous $\Rightarrow$ left continuous”

Consider $(u_i)_{i \geq 1}$ with $u_i < u$ and $\lim_{i \to \infty} u_i = u$.

Clearly, $\lim_{i \to \infty} t(u_i, v) \leq t(u, v)$.

Assume $\lim_{i \to \infty} t(u_i, v) < t(u, v)$.

Define $\varepsilon := (t(u, v) - \lim_{i \to \infty} t(u_i, v))/2$.

There exists $u_k$ with $t(u_k, v) > t(u, v) - \varepsilon$ $\rightarrow$ contradiction

Thus $\lim_{i \to \infty} t(u_i, v) = t(u, v)$ and $t$ is left continuous.
Left Continuity and Lower Semicontinuity (cont.)

“left continuous $\Rightarrow$ lower semicontinuous”

To show: $\forall \varepsilon > 0 : \exists \delta > 0 : \forall u' \in (u - \delta; u] : t(u', \nu) > t(u, \nu) - \varepsilon$

This is the definition of $\lim_{i \to \infty} t(u_i, \nu) = t(u, \nu)$.

Thus, $t$ lower semicontinuous.

Remark: Left continuousness equivalent to

$$t(\sup\{u_i\}, \nu) = \sup\{t(u_i, \nu)\}.$$

Remark 2: Right continuity and upper semicontinuity (and their equivalence for $t$-norms) can be proved the same way.
\( t\)-norms and \( \Phi\)-operators

**Theorem:** For a \( t\)-norm \( t \) there exists a \( \Phi\)-operator \( p_t \) connected with \( t \) iff \( t \) is lower semicontinuous.

**Proof:** “\( t \) lower semicontinuous \( \Rightarrow \) \( p_t \) is \( \Phi\)-operator”

need (1): \( v \leq w \Rightarrow p_t(u, v) \leq p_t(u, w) \)
\( \Leftrightarrow (v \leq w \Rightarrow \sup\{x \mid t(u, x) \leq v\} \leq \sup\{x \mid t(u, x) \leq w\}) \)
\( \checkmark \) since \( t \) non-decreasing

need (2): \( t(u, p_t(u, v)) \leq v \)
\( t(u, p_t(u, v)) = t(u, \sup\{w \mid t(u, w) \leq v\}) \) by definition
\( = \sup\{t(u, w) \mid t(u, w) \leq v\} \leq v \)

need (3): \( v \leq p_t(u, t(u, v)) \)
\( p_t(u, t(u, v)) = \sup\{w \mid t(u, w) \leq t(u, v)\} \) by definition
\( \geq v. \)
Proof: “$p_t$ is Φ-operator $\Rightarrow$ $t$ lower semicontinuous”
$\iff (t$ not lower semicontinuous $\Rightarrow p_t$ is no Φ-operator)

$t$ not lower semicontinuous
$\Rightarrow \exists u \in [0, 1], (v_i)_{i \geq 1}: \sup\{t(u, v_i)\} < t(u, \sup\{v_i\})$

Assume $p_t$ (Φ-operator connected with $t$) exists.

Let $v := \lim_{i \to \infty} v_i$:

- $t(u, p_t(u, v)) = t(u, \sup\{w \mid t(u, w) \leq v\})$ by definition of $p_t$
- $t(u, \sup\{w \mid t(u, w) \leq v\}) > \sup\{t(u, w) \mid t(u, w) \leq v\} = v$
- Hence, $t(u, p_t(u, v)) > v$

— Contradiction: $p_t$ is no Φ-operator.
Definition: For a lower semicontinuous $t$-norm $t$ we denote the uniquely determined $\Phi$-operator connected to $t$ by $\phi_t$, i.e.,
\[ \phi_t(u, v) = \sup \{ w \mid t(u, w) \leq v \} \]

What $\phi_t$ do we get for “our” $t$-norms?

minimum: $t_m(u, v) = \min\{u, v\}$

$\phi_{t_m}(u, v) = \sup\{w \mid \min\{u, w\} \leq v\} = \begin{cases} 1 & \text{if } u \leq v \\ v & \text{otherwise} \end{cases}$
Łukasiewicz: \( t_L(u, v) = \max\{0, u + v - 1\} \)

\( \phi_{t_L}(u, v) = \sup\{w \mid \max\{0, u + w - 1\} \leq v\} \)

= \min\{1, 1 - u + v\}

**product:** \( t_p(u, v) = u \cdot v \)

\( \phi_{t_p}(u, v) = \sup\{w \mid u \cdot w \leq v\} \)

= \begin{cases} 
\min\{1, v/u\} & \text{if } u \neq 0 \\
1 & \text{otherwise}
\end{cases} \)
$\phi_{tm}$ connected to minimum $t$-norm

$$\phi_{tm}(u, v) = \begin{cases} 
1 & \text{if } u \leq v \\
v & \text{otherwise}
\end{cases}$$
\( \phi_{t_L} \) connected to Łukasiewicz \( t \)-norm

Łukasiewicz: \( \phi_{t_L}(u, v) = \min\{1, 1 - u + v\} \)
$\phi_{tp}$ connected to product $t$-norm

\[
\phi_{tp}(u, v) = \begin{cases} 
\min\{1, v/u\} & \text{if } u \neq 0 \\
1 & \text{otherwise}
\end{cases}
\]
Properties of $\phi_t$

**Theorem:** Let $t$ be a t-norm. For each $\phi_t$ defined via

$$\phi_t(u, v) = \sup\{w \mid t(u, w) \leq v\}$$

the following holds:

1. $u \leq v \Rightarrow \phi_t(u, w) \geq \phi_t(v, w)$
2. $u \leq v \Rightarrow \phi_t(u, v) = 1$
3. $\phi_t(1, v) = v$
4. $v \leq \phi_t(u, v)$

**Proof:**

1. $u \leq v$: need $\sup \{x \mid t(u, x) \leq w\} \geq \sup \{x \mid t(v, x) \leq w\}$  \(\checkmark\)
2. $u \leq v$: need $\sup \{w \mid t(u, w) \leq v\} = 1$  \(\checkmark\)
3. need $\sup \{w \mid t(1, w) \leq v\} = v$  \(\checkmark\)
4. need $v \leq \sup \{w \mid t(u, w) \leq v\}$  \(\checkmark\)
More Properties of $\phi_t$

**Theorem:** For each $\phi_t$ defined via $\phi_t(u, v) = \sup\{w \mid t(u, w) \leq v\}$ for a lower semicontinuous $t$ the following holds:

1. $\phi_t(u, \phi_t(v, t(u, v))) = 1$
2. $\phi_t(t(u, v), w) \leq \phi_t(u, \phi_t(v, w))$
3. $\phi_t(u, v) \leq \phi_t(t(u, w), t(v, w))$

**Proof:**

1. $\phi_t(v, t(u, v)) = \sup\{w \mid t(v, w) \leq t(v, u)\} = u$; need $\phi_t(u, u) = 1$ \(\sqrt{\text{ }}\)
2. $\phi_t(t(u, v), w) = \sup\{x \mid t(t(u, v), x) \leq w\}$\[\phi_t(u, \phi_t(v, w)) = \sup\{x \mid t(u, x) \leq \phi_t(v, w)\}\]
   \[= \sup\{x \mid t(u, x) \leq \sup\{x' \mid t(v, x') \leq w\}\}\]
   \(\sqrt{\text{ }}\)
3. $\phi_t(u, v) = \sup\{x \mid t(u, x) \leq v\}$\[\phi_t(t(u, w), t(v, w)) = \sup\{x \mid t(t(u, w), x) \leq t(v, w)\}\]
   need: $\sup\{x \mid t(u, x) \leq v\} \leq \sup\{x \mid t(t(u, x), w) \leq t(v, w)\}$
   \(\sqrt{\text{ }}\)
Some More Properties of $\phi_t$

**Theorem:** For each $\phi_t$ defined via $\phi_t(u, v) := \sup\{w \mid t(u, w) \leq v\}$ for a lower semicontinuous $t$ the following holds:

1. $u \leq v \iff \phi_t(u, v) = 1$
2. $\phi_t(t(u, v), w) = \phi_t(u, \phi_t(v, w))$

**Proof:**

1. we have: $u \leq v \implies \phi_t(u, v) = 1$
2. we need: $u \leq v \iff \phi_t(u, v) = 1$
3. $1 = \phi_t(u, v) = \sup\{w \mid t(u, w) \leq v\}$
4. $\implies u = t(u, 1) \leq v$ \(\checkmark\)
Some More Properties of $\phi_t$

**Theorem:** For each $\phi_t$ defined via $\phi_t(u, v) := \sup \{w \mid t(u, w) \leq v\}$ for a lower semicontinuous $t$ the following hold:

1. $u \leq v \iff \phi_t(u, v) = 1$
2. $\phi_t(t(u, v), w) = \phi_t(u, \phi_t(v, w))$

**Proof:**

2. we have: $\phi_t(t(u, v), w) \leq \phi_t(u, \phi_t(v, w))$
   
   we need: $\phi_t(t(u, v), w) \geq \phi_t(u, \phi_t(v, w))$

   $\phi_t(u, \phi_t(v, w)) = \sup \{x \mid t(u, x) \leq \phi_t(v, w)\}$
   
   $\phi_t(t(u, v), w) = \sup \{x \mid t(t(u, v), x) \leq w\}$

   Thus, suffices: $t(u, x) \leq \phi_t(v, w) \Rightarrow t(u, t(v, x)) \leq w$

   $t(u, t(v, x)) = t(v, t(u, x)) \leq t(v, \phi_t(v, w)) \leq w$
Negation via Implication

Φ-operators yield interesting negations . . .

**Definition:** Let \( t \) be some lower semicontinuous \( t \)-norm. Define the negation connected to \( t \) as \( n_t(u) := \phi_t(u, 0) \)

**Theorem:** For any lower semicontinuous \( t \)-norm \( t \), \( n_t \) is a negation.

**Proof:** We need:

1. \( n_t(0) = 1: \phi_t(0, 0) = \sup\{w \mid t(0, w) \leq 0\} = 1 \checkmark \)
2. \( n_t(1) = 0: \phi_t(1, 0) = \sup\{w \mid t(1, w) \leq 0\} = 0 \checkmark \)
3. \( n_t \) non-increasing: \( n_t(u) = \phi_t(u, 0) = \sup\{w \mid t(u, w) \leq 0\} \checkmark \)
Concrete Negations via $\phi_t$

**minimum:** $\phi_{tm}(u, v) = \begin{cases} 1 & \text{if } u \leq v \\ v & \text{otherwise} \end{cases}$

$n_{tm}(u) = \phi_{tm}(u, 0) = \begin{cases} 1 & u = 0 \\ 0 & u \neq 0 \end{cases}$

**Łukasiewicz:** $\phi_{tl}(u, v) = \min\{1, 1 - u + v\}$

$n_{tl}(u) = \phi_{tl}(u, 0) = 1 - u$

**product:** $\phi_{tp}(u, v) = \begin{cases} \min\{1, v/u\} & \text{if } u \neq 0 \\ 1 & \text{otherwise} \end{cases}$

$n_{tp}(u) = \phi_{tp}(u, 0) = \begin{cases} 1 & u = 0 \\ 0 & u \neq 0 \end{cases}$
Motivation

Now we have

- fuzzy conjunction (via $t$-norms),
- fuzzy disjunction (via $t$-conorms),
- fuzzy implications (via $\Phi$-operators),
- fuzzy negations (via $\Phi$-operators), and
- fuzzy numbers.

What do we want to do now?

We want to model and control technical systems.

$\implies$ Fuzzy Control
Fuzzy Control

Control can often be described as a functional relationship $f : A \rightarrow B$.

We’d like to have “fuzzy functions.”

More general are fuzzy relations.

Crisp relations can be described as subsets.

We reconsider fuzzy sets, first.
Fuzzy Sets — Again

We already introduced fuzzy sets . . . but we did not work with them.

We want to have operations on fuzzy sets similar to operations on crisp sets.

- intersection $\cap$
- union $\cup$
- complement $^C$
- subsets $\subseteq$
Operations on Fuzzy Sets

Using crisp sets as bottom line:

**intersection:**

- **crisp:** \( A \cap B = \{ x \mid x \in A \land x \in B \} \)
- **fuzzy:** \( A \cap_t B = \{ x \mid t(x \in A, x \in B) \} \) for any \( t \)-norm \( t \), i.e. the fuzzy set \( A \cap_t B \) is defined as

\[
(A \cap_t B)(x) := t(A(x), B(x)).
\]
$t_m$-based Intersection

$t_m(u, v) = \min\{u, v\}$
$t_L$-based Intersection

$t_L(u, v) = \max\{0, u + v - 1\}$
$t_p$-based Intersection

$t_p(u, v) = uv$
$t_d$-based Intersection

\[ t_d(u, v) = \begin{cases} \min\{u, v\} & \text{if } \max\{u, v\} = 1 \\ 0 & \text{otherwise} \end{cases} \]
Operations on Fuzzy Sets

Using crisp sets as bottom line:

**union:**
- **crisp:** $A \cup B = \{x \mid x \in A \lor x \in B\}$
- **fuzzy:** $A \cup_t B = \{x \mid s_t(x \in A, x \in B)\}$ for any $t$-norm $t$, i.e. the fuzzy set $A \cup_t B$ is defined as

$$(A \cup_t B)(x) := s_t(A(x), B(x)).$$
$t_m$-based Union

$$s_{t_m}(u, v) = \max\{u, v\}$$
$t_L$-based Union

$$s_{t_L}(u, v) = \min\{1, u + v\}$$
$t_p$-based Union

$$s_{t_p}(u, v) = u + v - uv$$
$t_d$-based Union

$$s_{t_d}(u, v) = \begin{cases} 
\max\{u, v\} & \text{if } uv = 0 \\
1 & \text{otherwise}
\end{cases}$$
Using crisp sets as bottom line:

**complement:**

- **crisp:** $A^C = \{ x \mid \neg(x \in A) \}$
- **fuzzy:** $A^{C_n} = \{ x \mid n(x \in A) \}$ for any negation $n$

i.e. the fuzzy set $A^{C_n}$ is defined as

$$(A^{C_n})(x) := n(A(x)).$$
$n_L$-based Negation

$n_L(u) = 1 - u$
$n_d$-based Negation

\[ n_d(u) = \begin{cases} 
1 & \text{if } u = 0 \\
0 & \text{otherwise} 
\end{cases} \]
$n_{wd}$-based Negation

\[ n_{wd}(u) = \begin{cases} 
1 & \text{if } u \neq 1 \\
0 & \text{otherwise}
\end{cases} \]
Subsets

We already defined
\[ A \subseteq B \iff \forall x: A(x) \leq B(x). \]

Note: This is crisp!

Is it plausible?

Example:
\[ U = \mathbb{N}_0 \]
\[ A(x) := \begin{cases} \frac{1}{x^2} & \text{if } x > 0 \\ a & \text{if } x = 0 \end{cases} \quad B(x) := \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ b & \text{if } x = 0 \end{cases} \]

Clearly, \( \forall x \in U \setminus \{0\}: A(x) \leq B(x) \).

But because \( a > b \), \( \neg (A \subseteq B) \).
Fuzzy Subsets

A different perspective:
Crisp sets: \( A \subseteq B \iff \forall x: x \in A \Rightarrow x \in B \)

Definition:
For any lower semicontinuous \( t \)-norm \( t \):
\( A \subseteq_t B \iff \forall x \in U: \phi_t(A(x), B(x)) \)
\( A \equiv_t B \iff t(A \subseteq_t B, B \subseteq_t A) \)

What does \( \forall \) mean for fuzzy sets?

Definition:
\( \forall x: F(x) := \inf \{ F(u) \mid u \in U \} \)
\( \exists x: F(x) := \sup \{ F(u) \mid u \in U \} \)