On Graphs with Characteristic Bounded-Width Functions

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Abstract

Graphs can be represented symbolically by the Ordered Binary Decision Diagram (OBDD) of their characteristic Boolean function. The latter is said to have bounded width b if its minimal-size complete OBDD has bounded width b. Bounded-width functions have small OBDDs and allow efficient functional operations. Therefore, they are of interest in the analysis of symbolic graph algorithms. In this paper, a variety of graph sequences is proved to have characteristic functions of bounded-width. It is shown that this property is closed under important graph composition operations. Finally, an unpleasant property of a naive symbolic all-pairs shortest-paths algorithm in the context of bounded-width functions is presented.

1 Introduction

Algorithms on graphs G with node set V and edge set $E \subseteq V^2$ typically work on adjacency lists of size $\Theta(|V| + |E|)$ or on adjacency matrices of size $\Theta(|V|^2)$. These representations are called *explicit*. However, there are application areas in which problems on graphs of such large size have to be solved that an explicit representation on today's computers is not possible. In the area of logic synthesis and verification, state-transition graphs with for example 10^{27} nodes and 10^{36} edges occur. Other applications produce graphs which are representable in

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explicit form, but for which even runtimes of efficient polynomial algorithms are not practicable anymore. Modeling of the WWW, street, or social networks are examples of this problem scenario.

Symbolic graph representation. Yet, we expect the large graphs occurring in application areas to contain regularities. If we consider graphs as Boolean functions, we can represent them by *Ordered Binary Decision Diagrams* (*OBDDs*) [2, 3, 19]. This data structure is well established in verification and synthesis of sequential circuits [6, 7, 9, 10, 19] due to its good compression of regular structures. In order to represent a graph $G = (V, E)$ by an OBDD, its edge set E is considered as a *characteristic Boolean function* χ_E, which maps binary encodings of E's elements to 1 and all others to 0. This representation is called *implicit* or *symbolic*, and is not essentially larger than explicit ones. Nevertheless, we hope that advantageous properties of G lead to small, that is sublinear OBDD-sizes [16, 20].

Symbolic algorithms. Having such an OBDD-representation of a graph, we are interested in solving problems on it without extracting too much explicit information from it. Algorithms that are mainly restricted to the use of functional operations are called *implicit* or *symbolic algorithms* [4, 5, 8, 14, 15, 18, 19, 21, 22]. They are considered as heuristics to save time and/or space when large structured input graphs do not fit into the internal memory anymore. Then, we hope that each OBDD-operation processes many edges in parallel. The runtime of such methods depends on the number of executed operations as well as on the efficiency of each single one. The latter in turn depends on the size of the operand OBDDs.

Graphs with characteristic bounded-with functions. In general, it is desirable to show that symbolic algorithms behave efficient w. r. t. the sizes of their input OBDDs. In particular, this requires that the computed Boolean output functions have small OBDD. This task is difficult in general, because even a number of $\Theta(\log |V|)$ OBDD-operations may suffice to cause an exponential blow-up of the OBDD-size. In most papers on symbolic algorithms, their usability is just proved by experiments on benchmark inputs from special application areas [8, 9, 10, 12, 22]. In other works considering more general graph problems, mostly the number of OBDD-operations (often referred to as "symbolic steps") is bounded as a hint on the actual runtime [1, 4, 5, 13].

The restricted class of Boolean bounded-width functions has small OBDDs and allows efficient OBDD-operations. Its convenient properties are used in some analyses of symbolic algorithms. In [16] and [21], grid graphs are considered, whereas [17] addresses the OBDD-size of time-expanded networks. In [18] a symbolic all-pairs shortest-paths algorithm is analyzed on general graph sequences with characteristic bounded-width functions. It is proved that the algorithm has polynomial runtime and space usage w. r. t. to the input graph's OBDD-size if both input OBDD and output OBDD represent bounded-width functions.

This paper gives proofs that have been omitted in [18] due to the lack of space. Moreover, various graphs are proved to have characteristic bounded-width functions in order to emphasize the representational power of this restricted function class.

Contents. This paper is organized as follows: Section 2 gives foundations on symbolic graph representations by OBDDs and introduces some preliminaries. In Sect. 3, bounded-width functions and their convenient properties are discussed. These are used in Sect. 4 to show the closedness of the bounded-width property under important graph composition operations. Section 5 shows for basic graph sequences that they have characteristic bounded-width functions. The latter is also the case for threshold graphs having certain properties, which are presented in Sect. 6. Finally, Sect. 7 gives a reason for the restriction to strictly positive edge weights in [18]; that is, a special graph sequence with unpleasant properties is constructed.

2 Preliminaries

We denote the class of Boolean functions $f\colon \{0,1\}^n \to \{0,1\}$ by B_n. The ith character of a binary number $x \in \{0,1\}^n$ is denoted by x_i and $|x| := \sum_{i=0}^{n-1} x_i 2^i$ identifies its value.

Consider a directed graph $G = (V, E)$ with node set $V = \{v_0, \ldots, v_{N-1}\}$ and edge set $E \subseteq V^2$. G can be represented by a *characteristic* Boolean function $\chi_E \in B_{2n}$ which maps pairs $(x, y) \in \{0,1\}^{2n}$ of binary node numbers of length $n := \lceil \log N \rceil$ to 1 iff $(v_{|x|}, v_{|y|}) \in E$. We can capture more complex graph properties by adding further arguments to characteristic functions. An additional weight function $c\colon E \to \{0, \ldots, 2^m - 1\}$ is modeled by $\chi_c \in B_{2n+m}$ which maps triples (x, y, d) to 1 iff $(v_{|x|}, v_{|y|}) \in E$ and $c(v_{|x|}, v_{|y|}) = |d|$.

Dependent of the binary node encoding, symbolically represented graphs may contain redundant singletons. For example, if N is not a power of two, χ_E is implicitly defined on $2^n - N$ nodes not contained in the original set V. Although this has no effect on many problems like maximum flows or shortest paths, some applications may require to distinguish between real nodes and redundant ones. Then, a characteristic function $\chi_{VALID} \in B_n$ can be used which maps node numbers x to 1 iff $v_{|x|} \in V$ (see Section 3.2).

Ordered Binary Decision Diagrams (OBDDs). A Boolean function $f \in B_n$ defined on variables $x_0, \ldots, x_{n-1}$ can be represented by an *Ordered Binary Decision Diagram* (*OBDD*) [2, 3, 19]. An OBDD $\mathcal{G}$ is a directed acyclic graph consisting of *internal nodes* and *sink nodes*. Each internal node is labeled with a

Boolean variable x_i, while each sink node is labeled with a Boolean constant. Each internal node is left by two edges one labeled by 0 and the other by 1. A *function pointer* p marks a special node that represents f. Moreover, a permutation $\pi \in \Sigma_n$ called the *variable order* must be respected by the internal nodes' labels on every path from p to a sink. For a given variable assignment $a \in \{0,1\}^n$, we compute the function value $f(a)$ by traversing $\mathcal{G}$ from p to a sink labeled with $f(a)$ while leaving a node x_i via its a_i-edge.

An OBDD $\mathcal{G}$ with variable order π is called π-OBDD. Its size size($\mathcal{G}$) is measured by the number of its nodes. The minimal-size π-OBDD for a function $f \in B_n$ is known to be canonical and will be denoted by $\pi\mathcal{G}[f]$ in this paper. We adopt the usual assumption that all OBDDs occurring in symbolic algorithms have minimal size, since all essential OBDD-operations produce minimized diagrams. There is an upper bound of $\big(2+o(1)\big)2^n/n$ for the OBDD-size of every $f \in B_n$; hence, an edge set $E \subseteq V^2$ has worst-case OBDD-size $\mathcal{O}(V^2/\log|V|)$.

The satisfiability of f can be decided in time $\mathcal{O}(1)$. The negation $\overline{f}$ as well as the replacement of a function variable x_i by a constant c (i. e., $f_{|x_i=c}$) is obtained in time $\mathcal{O}\big(\text{size}(\pi\mathcal{G}[f])\big)$ without enlarging the OBDD. Whether two functions f and g are equivalent (i. e., $f = g$) can be decided in time $\mathcal{O}\big(\text{size}(\pi\mathcal{G}[f]) + \text{size}(\pi\mathcal{G}[g])\big)$. These operations are called *cheap*. Further essential operations are the *binary synthesis* $f \otimes g$ for $f, g \in B_n$, $\otimes \in B_2$ (e. g., "$\wedge$" and "$\vee$"), and the *quantification* $(\mathcal{Q}x_i)f$ for a quantifier $\mathcal{Q} \in \{\exists, \forall\}$. In general, the result $\pi\mathcal{G}[f \otimes g]$ has size $\mathcal{O}\big(\text{size}(\pi\mathcal{G}[f]) \cdot \text{size}(\pi\mathcal{G}[g])\big)$, which is also the general runtime of this operation. The computation of $\pi\mathcal{G}\big[(\mathcal{Q}x_i)f\big]$ can be realized by two cheap operations and one binary synthesis in time and space $\mathcal{O}\big(\text{size}^2(\pi\mathcal{G}[f])\big)$.

Notation of characteristic functions. The characteristic functions used for symbolic representation are typically defined on a number of k subsets of Boolean variables, each representing a different argument (e. g., $C(x, y, d)$ is defined on nodes x, y and weight d). We assume w. l. o. g. that all arguments consist of the same number of n Boolean variables. If there is no confusion, both a function $\chi_S \in B_{kn}$ defined on $x^{(1)}, \ldots, x^{(k)} \in \{0,1\}^n$ as well as its OBDD-representation $\pi\mathcal{G}[\chi_S]$ will be denoted by $S(x^{(1)}, \ldots, x^{(k)})$ in this paper. Quantifications $(\mathcal{Q}x_0^{(i)}, \ldots, x_{n-1}^{(i)})$ over all n variables of argument i will be denoted by $(\mathcal{Q}x^{(i)})$.

Interleaved variable orders. Assume that each of the k function arguments $x^{(1)}, \ldots, x^{(k)} \in \{0,1\}^n$ has its own variable order $\tau_i \in \Sigma_n$. The global order π is called *k-interleaved* if it respects each τ_i while reading variables $x_j^{(i)}$ with same bit index j en bloc, that is, $\pi := (x^{(1)}_{\tau_1(0)}, x^{(2)}_{\tau_2(0)}, \ldots, x^{(k)}_{\tau_k(0)}, x^{(1)}_{\tau_1(1)}, \ldots, x^{(k)}_{\tau_k(n-1)})$. We say that *argument i is read with increasing bit significance* if $\tau_i = \text{id}$.

Using π enables to *swap* [19] the variables of two arguments $x^{(i)}$ and $x^{(i+1)}$ in time $\mathcal{O}\big(\text{size}(\pi\mathcal{G}[f])\big)$ resulting in a π-OBDD of worst-case size $3 \cdot \text{size}(\pi\mathcal{G}[f])$

(e. g., $S'(x, y) := S(y, x)$). The same bound holds for the resulting OBDD-size of a sequence of ℓ swaps moving argument $x^{(i)}$ to position $j = i + \ell$.

Definition 1 *Let $\rho \in \Sigma_k$ and $f \in B_{kn}$ be defined on variables $x^{(1)}, \dots, x^{(k)} \in \{0,1\}^n$. The argument reordering operation $\mathcal{R}_\rho \colon B_{kn} \to B_{kn}$ is defined by $\mathcal{R}_\rho\big(f(x^{(1)}, \dots, x^{(k)})\big) := f(x^{(\rho(1))}, \dots, x^{(\rho(k))})$.*

An argument reordering can be achieved by at most k swap sequences each not longer than k. Hence, the OBDD $\pi\mathcal{G}[\mathcal{R}_\rho(f)]$ has worst-case size $3^k \cdot \text{size}(\pi\mathcal{G}[f])$ and can be computed in time and space $\mathcal{O}\big(k^3 3^k \cdot \text{size}(\pi\mathcal{G}[f])\big)$. Because k is independent of f, this is considered as linear in $\text{size}(\pi\mathcal{G}[f])$.

3 Bounded-Width Functions

We introduce the class of Boolean bounded-width functions, whose convenient properties have been successfully used in [16, 17, 18, 21] to analyze symbolic algorithms.

Definition 2 *A π-OBDD for a function $f \in B_n$ is called* complete *if every path from its function pointer to a sink has length n.*

That is, complete OBDDs are not allowed to skip variable tests. The minimal-size complete π-OBDD for $f \in B_n$ is also known to be canonical [19] and will be denoted by $\pi\mathcal{G}_c[f]$ in the following.

Definition 3 *Let $F := (f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n \in B_{\mathcal{N}(n)}$, $\mathcal{N} \colon \mathbb{N} \to \mathbb{N}$, defined on variables $x_0, \dots, x_{\mathcal{N}(n)-1}$. Moreover, let $\Pi := (\pi_n)_{n\in\mathbb{N}}$ be a sequence of variable orders $\pi_n \in \Sigma_{\mathcal{N}(n)}$. F has* bounded width b w. r. t. Π *(F is* b-bounded by Π*) iff for all $n \in \mathbb{N}$ the OBDD $\pi_n\mathcal{G}_c[f_n]$ contains no more than b nodes labeled with the same variable x_i for $i \in \{0, \dots, \mathcal{N}(n) - 1\}$.*

Note that $\pi_n\mathcal{G}[f_n] \le \pi_n\mathcal{G}_c[f_n] = \mathcal{O}\big(\mathcal{N}(n)b\big)$. Both negation and variable replacement by constants do not increase the OBDD-size. We now prove that bounded-width functions are also closed under the OBDD-operations mentioned in Sect. 2 which may increase the OBDD-size: Binary synthesis, quantification, and argument reordering.

Let $F^{(1)} := (f_n^{(1)})_{n\in\mathbb{N}}$ and $F^{(2)} := (f_n^{(2)})_{n\in\mathbb{N}}$ be sequences of functions $f_n^{(1)}, f_n^{(2)} \in B_{\mathcal{N}(n)}$, $\mathcal{N} \colon \mathbb{N} \to \mathbb{N}$, defined on variables $x_0, \dots, x_{\mathcal{N}(n)-1}$. Assume $F^{(1)}$ and $F^{(2)}$ to have bounded width b_1 resp. b_2 w. r. t. variable orders $\Pi := (\pi_n)_{n\in\mathbb{N}}$, $\pi_n \in \Sigma_{\mathcal{N}(n)}$.

Theorem 1 *For all $n \in \mathbb{N}$, the OBDD $\pi_n\mathcal{G}[f_n^{(1)} \otimes f_n^{(2)}]$, $\otimes \in B_2$, can be computed in time and space $\mathcal{O}\big(\mathcal{N}(n)b_1b_2\big)$. The resulting sequence $(f_n^{(1)} \otimes f_n^{(2)})_{n\in\mathbb{N}}$ is b_1b_2-bounded by Π.*

Proof. We adopt the usual assumption that the binary synthesis is computed by recursively constructing the reachable subgraph of the product graph of $\pi_n\mathcal{G}[f_n^{(1)}]$ and $\pi_n\mathcal{G}[f_n^{(2)}]$ before minimizing the result in linear time [19]. This constructed part of the product graph will be denoted by $\pi_n PROD[f_n^{(1)}, f_n^{(2)}]$. Moreover, we assume w. l. o. g. $\pi_n = \mathrm{id}$.

At first, we prove the theorem's statements for the computation of $\mathrm{id}\mathcal{G}_\mathrm{c}[f_n^{(1)} \otimes f_n^{(2)}]$ from $\mathrm{id}\mathcal{G}_\mathrm{c}[f_n^{(1)}]$ and $\mathrm{id}\mathcal{G}_\mathrm{c}[f_n^{(2)}]$. Consider OBDD-nodes v_1 of $\mathrm{id}\mathcal{G}_\mathrm{c}[f_n^{(1)}]$ resp. v_2 of $\mathrm{id}\mathcal{G}_\mathrm{c}[f_n^{(2)}]$ representing subfunctions $g^{(1)}$ resp. $g^{(2)}$. Due to both OBDDs being complete, $g^{(1)}$ and $g^{(2)}$ participate in a recursive computation $g^{(1)} \otimes g^{(2)}$ if and only if v_1 and v_2 are labeled with the same Boolean variable x_i. Hence, the reachable subgraph $\mathrm{id}PROD_\mathrm{c}[f_n^{(1)}, f_n^{(2)}]$ of the complete product graph contains at most $b_1 b_2$ nodes labeled with x_i and is computed in time and space $\mathcal{O}\big(\mathcal{N}(n) b_1 b_2\big)$.

Now we investigate the binary synthesis of minimal-size OBDDs $\mathrm{id}\mathcal{G}[f_n^{(1)}]$ and $\mathrm{id}\mathcal{G}[f_n^{(2)}]$. In general, $\mathrm{id}PROD[f, g]$ may have worst-case size $\Omega\big(\mathrm{size}(\pi\mathcal{G}[f]) \cdot \mathrm{size}(\pi\mathcal{G}[g])\big)$, because recursive computations may combine nodes with different variable labels. This is not the case for $f_n^{(1)} \otimes f_n^{(2)}$.

Consider OBDD-nodes v_1 of $\mathrm{id}\mathcal{G}[f_n^{(1)}]$ resp. v_2 of $\mathrm{id}\mathcal{G}[f_n^{(2)}]$ representing subfunctions $g^{(1)}$ resp. $g^{(2)}$. Let v_1 be labeled by x_i and v_2 labeled by x_j with $i \le j$. If $\mathrm{id}PROD[f_n^{(1)}, f_n^{(2)}]$ contains a product node (v_1, v_2), then $\mathrm{id}PROD_\mathrm{c}[f_n^{(1)}, f_n^{(2)}]$ contains a node (v_1, v_2') which also represents $g^{(1)} \otimes g^{(2)}$. v_2' represents $g^{(2)}$ and is labeled by x_i, too. Because every pair $(g^{(1)}, g^{(2)})$ of subfunctions of $f_n^{(1)}$ and $f_n^{(2)}$ is represented at most once in $\mathrm{id}PROD[f_n^{(1)}, f_n^{(2)}]$, there is an injective map from the nodes of $\mathrm{id}PROD[f_n^{(1)}, f_n^{(2)}]$ to the nodes of $\mathrm{id}PROD_\mathrm{c}[f_n^{(1)}, f_n^{(2)}]$ implying that also $\mathrm{id}PROD[f_n^{(1)}, f_n^{(2)}]$ has size $\mathcal{N}(n) b_1 b_2$. □

Theorem 2 *Let $X := (X_n)_{n\in\mathbb{N}}$ be a sequence of variable sets $X_n \subseteq \big\{x_0, \dots, x_{\mathcal{N}(n)-1}\big\}$. For all $n \in \mathbb{N}$, the OBDD $\pi_n\mathcal{G}\big[(\mathcal{Q}X_n) f_n^{(1)}\big]$, $\mathcal{Q} \in \{\exists, \forall\}$, can be computed in time and space $\mathcal{O}\big(|X_n| \mathcal{N}(n) 2^{2b_1}\big)$. The resulting sequence $\big((\mathcal{Q}X_n) f_n^{(1)}\big)_{n\in\mathbb{N}}$ is 2^{b_1}-bounded by Π.*

Proof. Let $X_n = \{x_{i_1}, \dots, x_{i_r}\}$ and assume w. l. o. g. $\pi = \mathrm{id}$. Let $\mathcal{V}_i$ denote the set of OBDD-nodes of $\pi_n\mathcal{G}_\mathrm{c}[f_n^{(1)}]$ labeled with variable x_i. We construct an OBDD $\mathcal{G}^*$ (not of minimal-size) that also represents $f_n^{(1)}$: $\mathcal{G}^*$ consists of the node set $\mathcal{V}^* := \bigcup_{0\le i<\mathcal{N}(n)} \mathcal{P}(\mathcal{V}_i)$, where $v \in \mathcal{P}(\mathcal{V}_i)$ is labeled by x_i. Moreover, we define the a-successor $v(a)$, $a \in \{0, 1\}$, of a node $v \in \mathcal{V}^*$ by $v(a) := \bigcup_{w\in v}\{w(a)\}$. For v^* being the unique node of $\pi_n\mathcal{G}_\mathrm{c}[f_n^{(1)}]$ labeled by x_0, the function pointer of $\mathcal{G}^*$ points to $\{v^*\}$.

Let $\mathcal{G}_j^*$ be the OBDD for the intermediate quantification result $(\mathcal{Q}x_{i_j} \dots x_{i_1}) f_n^{(1)} =: g^{(j)}$. We perform the quantification over x_{i_j} by replacing all edges (u, v) by $\big(u, v(0) \cup v(1)\big)$ for all nodes $v \in \mathcal{P}(\mathcal{V}_{i_j})$. If the evaluation traversal on $\mathcal{G}_j^*$ visits $v(0) \cup v(1)$, this node represents all nodes reachable in the original OBDD $\pi_n \mathcal{G}_\text{c}[f_n^{(1)}]$ for any assignment of the quantified variables $x_{i_1}, \ldots, x_{i_j}$.

In generating $\mathcal{G}^*$ and $\mathcal{G}_1^*, \ldots, \mathcal{G}_\text{r}^*$, we also create terminal subsets of $\{0, 1\}$. When applying a sequence of existential quantifiers ($\mathcal{Q} = \exists$), the sets $\{1\}$ and $\{0, 1\}$ are replaced by the terminal 1 and all others by 0—it suffices if one assignment of the quantified variables leads to 1. When applying universal quantifiers ($\mathcal{Q} = \forall$), only the set $\{1\}$ is replaced by 1—all assignments must lead to 1.

We have constructed a complete OBDD $\mathcal{G}_r^*$ for $(\mathcal{Q}X_n) f_n^{(1)}$ of width 2^{b_1} implying the same width bound for the minimal-size complete OBDD $\text{id}\mathcal{G}\big[(\mathcal{Q}X_n) f_n^{(1)}\big]$. This result is generated by $r = |X_n|$ quantification operations $(\mathcal{Q}x_{i_j})$ each involving one binary synthesis $g_{|x_{i_j}=0}^{(j-1)} \otimes g_{|x_{i_j}=1}^{(j-1)}$ [19]. Due to $g^{(j-1)}$ having bounded width 2^{b_1}, each binary synthesis takes time and space $\mathcal{O}\big(\mathcal{N}(n) 2^{2b_1}\big)$. □

Let $F^{(3)} := (f_n^{(3)})_{n \in \mathbb{N}}$ be a sequence of functions $f_n^{(3)} \in B_{k\mathcal{N}(n)}$, $k \in \mathbb{N}$, $\mathcal{N} \colon \mathbb{N} \to \mathbb{N}$, defined on variables $x^{(1)}, \ldots, x^{(k)} \in \{0, 1\}^{\mathcal{N}(n)}$. Assume $F^{(3)}$ to have bounded width b_3 w. r. t. k-interleaved variable orders $\Pi := (\pi_n)_{n \in \mathbb{N}}$, $\pi_n \in \Sigma_{k\mathcal{N}(n)}$. Let $\rho \in \Sigma_k$.

Theorem 3 *For all $n \in \mathbb{N}$, the OBDD $\pi_n \mathcal{G}\big[\mathcal{R}_\rho(f_n^{(3)})\big]$ can be computed in time and space $\mathcal{O}\big(\mathcal{N}(n) b_3 k^3 2^k\big)$. The resulting sequence $\big(\mathcal{R}_\rho(f_n^{(3)})\big)_{n \in \mathbb{N}}$ is $b_3 2^k$-bounded by Π.*

Proof. Each argument $x^{(i)}$, $i \in \{1, \ldots, k\}$, can be brought to its new position $\rho^{-1}(i)$ by a sequence of at most $k - 1$ swap operations. At first, we consider $\pi_n \mathcal{G}_\text{c}[f_n^{(3)}]$: We assume w. l. o. g. that $\rho^{-1}(i) > i$. That is, the swap sequences moves argument $x^{(i)}$ to a higher position. Due to the k-interleaved variable order π_n, such a “jump-down” swap sequence shorter than k causes at most a doubling of the complete OBDD’s width [19]. Hence, reordering all k arguments results in a final width of at most $b_3 2^k$. This bound holds in particular for the minimal-size OBDD $\pi_n \mathcal{G}\big[\mathcal{R}_\rho(f_n^{(3)})\big]$.

Each of the $\mathcal{O}(k^2)$ swap operations is done in linear time and space w. r. t. its result of size $\mathcal{O}\big(k\mathcal{N}(n) b_3 2^k\big)$ implying the over-all resource bound of $\mathcal{O}\big(\mathcal{N}(n) b_3 k^3 2^k\big)$. □

The resulting width bounds are worst cases. However, because b_1, b_2, b_3, and k are independent of n, each operation takes linear time and space w. r. t. the number $\mathcal{N}(n)$ of variables. We conclude that bounded-width functions are closed under all operations introduced in Sect. 2.

3.1 Multivariate Threshold Functions.

We will use comparisons like $F(x, y, z) := (|x| + |y| = |z|)$ as building blocks of characteristic functions. These can be composed of *multivariate threshold functions.*

Definition 4 (Woelfel [21]) *Let $f \in B_{kn}$ be defined on variables $x^{(1)}, \ldots, x^{(k)} \in \{0,1\}^n$. Moreover, let W, $T \in \mathbb{Z}$, and $w_1, \ldots, w_k \in \{-W, \ldots, W\}$. f is called* k-variate threshold function *iff*

$$f(x^{(1)}, \ldots, x^{(k)}) = \left(\sum_{i=1}^{k} w_i \cdot |x^{(i)}| \geq T \right) .$$

W is called the maximum absolute weight *of f. The class of k-variate threshold functions $f \in B_{kn}$ with maximum absolute weight W is denoted by $\mathbb{T}_{k,n}^W$.*

Obviously, F can be expressed as $(|x| + |y| - |z| \geq 0) \wedge (|z| - |x| - |y| \geq 0)$. Analogue, the relations $>$, $\leq$, and $<$ can be composed of multivariate threshold functions, too.

Theorem 4 (Woelfel [21]) *Let $F := (f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n \in B_{k\mathcal{N}(n)}$, $k \in \mathbb{N}$, $\mathcal{N}\colon \mathbb{N} \to \mathbb{N}$, and $\Pi := (\pi_n)_{n\in\mathbb{N}}$ k-interleaved variable orders $\pi_n \in \Sigma_{k\mathcal{N}(n)}$ with increasing bit significance. If for all $n \in \mathbb{N}$ it is $f_n \in \mathbb{T}_{k,\mathcal{N}(n)}^W$ then F is $\mathcal{O}(k^2 W)$-bounded by Π.*

From the closedness under binary synthesis, we conclude that weighted comparisons with relations $>$, $\geq$, $=$, $<$, and $\leq$ have bounded width for k-interleaved variable orders Π with increasing bit significance. This fact can be used to show bounded width for the characteristic functions of various basic graphs (see Sect. 5). Because W and k are independent of n, the comparisons have π_n-OBDD-size $\mathcal{O}\big(\mathcal{N}(n)\big)$.

3.2 Symbolic Representation of Valid Node Numbers

Consider a graph $G = (V, E)$ whose symbolic representation $E(x, y)$ contains more than $N := |V|$ nodes. This may have various reasons: N might be not a power of two. Alternatively, the existence of redundant nodes may ease the encoding of G's nodes. Then, a characteristic function $\mathit{VALID}(x)$ can be used to represent all valid node numbers x with $v_{|x|} \in V$.

Consider a graph sequence $G = \big(G_n = (V_n, E_n)\big)_{n\in\mathbb{N}}$ with $N_n := |V_n|$ and its symbolic counterpart $E(x, y)_n$. We assume the following node numbering scheme: Each binary node number x consists of $\ell\mathcal{N}(n)$ Boolean variables, $\mathcal{N}\colon \mathbb{N} \to \mathbb{N}$, representing ℓ subnumbers $s_1(x), \ldots, s_\ell(x) \in \{0,1\}^{\mathcal{N}(n)}$ with $s_i(x) := x_{(\ell-i+1)\mathcal{N}(n)-1} \ldots x_{(\ell-i)\mathcal{N}(n)}$. Symbolic algorithms are typically working on characteristic functions $f \in B_{k\ell\mathcal{N}(n)}$ defined on a number of k node arguments $x^{(1)}, \ldots, x^{(k)} \in \{0,1\}^{\ell\mathcal{N}(n)}$ using k-interleaved variable order $\Pi := (\pi_n)_{n\in\mathbb{N}}$, $\pi_n \in \Sigma_{k\ell\mathcal{N}(n)}$. As defined in Sect. 2, each argument $x^{(i)}$ is read according to some local order $\tau_{i,n} \in \Sigma_{\ell\mathcal{N}(n)}$. In order to enable that each subnumber $s_j(x^{(i)})$ is an argument of a multivariate threshold function, we choose ℓ-interleaved variable orders $\tau_{i,n}$ for $i \in \{1, \ldots, k\}$. Each $\tau_{i,n}$ respects local orders $\phi_{i,j,n} \in \Sigma_{\mathcal{N}(n)}$ for the $\mathcal{N}(n)$ bits of the ℓ subnumbers, that is,

$$\tau_{i,n} := \big(s_1(x^{(i)})_{\phi_{i,1,n}(0)}, s_2(x^{(i)})_{\phi_{i,2,n}(0)}, \ldots, s_\ell(x^{(i)})_{\phi_{i,\ell,n}(0)}, \\ s_1(x^{(i)})_{\phi_{i,1,n}(1)}, \ldots, s_\ell(x^{(i)})_{\phi_{i,\ell,n}(\mathcal{N}(n)-1)}\big) \ .$$

Obviously, π_n is $k\ell$-interleaved, too.

We consider two possible terms of valid nodes.

Componentwise thresholds. At first, a binary node number $x \in \{0,1\}^{\ell\mathcal{N}(n)}$ might be valid iff the value $|s_j(x)|$ of each number-component $s_j(x)$ is smaller than some threshold $T_{j,n}$ for $j \in \{1, \ldots, \ell\}$. The corresponding functions $\mathit{VALID} := \big(\mathit{VALID}(x)_n\big)_{n\in\mathbb{N}}$ can be expressed as

$$\mathit{VALID}(x)_n := \bigwedge_{j=1}^{\ell} |s_j(x)| < T_{j,n} \ .$$

This function is composed of ℓ comparisons. Due to Theorems 1 and 4, *VALID* is $\mathcal{O}(a^\ell)$-bounded by Π for some appropriate constant a if each order $\phi_{i,j,n}$ reads the $\mathcal{N}(n)$ bits of $s_j(x^{(i)})$ with increasing significance (i.e., $\phi_{i,j,n} = \mathrm{id}$).

Global thresholds. Secondly, *VALID* might be defined by $\mathit{VALID}(x)_n := (|x| < T_n)$ for threshold values $T_n \in \mathbb{N}$. This comparison has to be realized by comparisons of the subnumbers $s_1(x), \ldots, s_\ell(x)$ with corresponding parts of the binary representation of T_n. Let $T_{i,n} := \lceil T_n/2^{(\ell-i)\mathcal{N}(n)} \rceil \bmod 2^{(i-1)\mathcal{N}(n)}$.

$$\mathit{VALID}(x)_n = \bigvee_{j=1}^{\ell} \left(\big(|s_j(x)| < T_{j,n}\big) \wedge \bigwedge_{j'=1}^{j-1} \big(|s_{j'}(x)| = T_{j',n}\big) \right)$$

This function is composed of $\mathcal{O}(\ell^2)$ comparisons. Due to Theorems 1 and 4, *VALID* is $\mathcal{O}(a^{\ell^2})$-bounded by Π for some appropriate constant a if each order $\phi_{i,j,n}$ reads the $\mathcal{N}(n)$ bits of $s_j(x^{(i)})$ with increasing significance.

Remark 1 *If a threshold T is a power of 2, the comparisons correspond to checking if $|s_i(x)_{\mathcal{N}-1} \ldots s_i(x)_{\log T}| = 0$. This check is $\mathcal{O}(1)$-bounded by every variable order. Therefore, the assumption that all arguments of a characteristic function consist of the same number of Boolean variables is no essential limitation.*

4 Graph Composition Operations

Having the results on bounded-with functions, we ask what kinds of graphs can be represented by them. Therefore, we investigate the closedness of graphs having characteristic bounded-width functions under important graph composition operations which can be used to build complex graphs from basic ones.

In particular, we consider sequences $G = (G_n)_{n\in\mathbb{N}}$ of loopless directed graphs $G_n = (V_n, E_n, c_n)$ with node set V_n of size $N_n := |V_n|$, edge set $E_n \subseteq V_n^2$, and strictly positive edge weights $c_n\colon E_n \to \mathbb{N}_{>0}$. The results can be easily transferred to the case of graphs without weights. G_n has the symbolic representation $C(x, y, d)_n$ with

$$C(x,y,d)_n = 1 :\Leftrightarrow \big[(v_{|x|}, v_{|y|}) \in E_n\big] \wedge \big[c_n(v_{|x|}, v_{|y|}) = |d|\big] \ .$$

Moreover, we consider the sequence $(\mathrm{dist}_n)_{n\in\mathbb{N}}$ of functions $\mathrm{dist}_n\colon V_n^2 \to \mathbb{N}_0 \cup \{\infty\}$ mapping node pairs (u, v) to the length $\|\bar{p}\| := \sum_{e\in\bar{p}} c(e)$ of shortest u–v-paths $\bar{p} = (u, \dots, v)$. The corresponding characteristic function $DIST(x, y, d)_n$ is defined by

$$DIST(x,y,d)_n = 1 :\Leftrightarrow \mathrm{dist}_n(v_{|x|}, v_{|y|}) = |d| \ .$$

The maximum path length in G_n is bounded by $B_n(N_n-1)$. Let $B_n := \max\{c(e) \mid e \in E_n\}$ and $\mathcal{N}(n) = \Theta\big(\log(N_nB_n)\big)$ be the number of bits encoding one node number $|x|$ or distance value $|d| \leq B_n(N_n - 1)$ of G_n. From the closedness of bounded-width functions under OBDD-operations we now conclude the closedness under graph composition operations.

Let $G^{(i)} := \big(G_n^{(i)} = (V_n^{(i)}, E_n^{(i)}, c_n^{(i)})\big)_{n\in\mathbb{N}}$, $i \in \{1, 2, 3\}$, be sequences of loopless directed weighted graphs with same notation as G and $V_n^{(1)} \cap V_n^{(2)} = \emptyset$ for all $n \in \mathbb{N}$. Let $\Pi^{(i)} := (\pi_n^{(i)})_{n\in\mathbb{N}}$, $i \in \{1, 2\}$, be sequences of k-interleaved variable orders $\pi_n^{(i)} \in \Sigma_{k\mathcal{N}^{(i)}(n)}$.

Definition 5 *Graph Composition Operations.*

1. *$G^{(3)}$ is called the* cojoin *of $G^{(1)}$ and $G^{(2)}$ iff for all $n \in \mathbb{N}$ it is $V_n^{(3)} = V_n^{(1)} \cup V_n^{(2)}$, $E_n^{(3)} = E_n^{(1)} \cup E_n^{(2)}$, and $c_n^{(3)}(e) = c_n^{(i)}(e)$ for $e \in E_n^{(i)}$.*

2. *$G^{(3)}$ is called the $\mathcal{A}$*-join *of $G^{(1)}$ and $G^{(2)}$, $\mathcal{A}\colon \mathbb{N} \to \mathbb{N}_{>0}$, iff for all $n \in \mathbb{N}$ it is $V_n^{(3)} = V_n^{(1)} \cup V_n^{(2)}$, $E_n^{(3)} = E_n^{(1)} \cup E_n^{(2)} \cup (V_n^{(1)} \times V_n^{(2)})$, and $c_n^{(3)}(e) = c_n^{(i)}(e)$ for $e \in E_n^{(i)}$ resp. $c_n^{(3)}(e) = \mathcal{A}(n)$ for $e \in V_n^{(1)} \times V_n^{(2)}$.*

3. *$G^{(3)}$ is called the* node substitution *of $G^{(1)}$ in $G^{(2)}$ iff for all $n \in \mathbb{N}$ it is $V_n^{(3)} = V_n^{(1)} \times V_n^{(2)}$,*

$$E_n^{(3)} = \Big\{\big((t,u),\ (v,w)\big) \mid \big((t,v) \in E_n^{(1)} \wedge (u = w)\big) \vee (u,w) \in E_n^{(2)}\Big\} \ ,$$

and $c_n^{(3)}$ weights edge $\big((t,u),\ (v,w)\big)$ with $c_n^{(1)}(t,v)$ if $u = w$ resp. $c_n^{(2)}(u,w)$ if $(u,w) \in E_n^{(2)}$.

4. *$G^{(3)}$ is called the* product *of $G^{(1)}$ and $G^{(2)}$ iff for all $n \in \mathbb{N}$ it is $V_n^{(3)} = V_n^{(1)} \times V_n^{(2)}$,*

$$E_n^{(3)} = \Big\{\big((t,u),\ (v,w)\big) \mid \big((t,v) \in E_n^{(1)} \wedge (u = w)\big) \vee \big((t = v) \wedge (u,w) \in E_n^{(2)}\big)\Big\} \ ,$$

and $c_n^{(3)}$ *weights edge* $\big((t,u),\,(v,w)\big)$ *with* $c_n^{(1)}(t,v)$ *if* $u = w$ *resp.* $c_n^{(2)}(u,w)$ *if* $t = v$.

In order to prove properties of the symbolic results of graph composition operations, we first have to discuss how to encode the resulting nodes $V_n^{(3)}$.

4.1 Union Compositions

At first, we consider the case $V_n^{(3)} = V_n^{(1)} \cup V_n^{(2)}$. Without limitation of generality, we assume the same number $\mathcal{N}(n) := \mathcal{N}^{(1)}(n) = \mathcal{N}^{(2)}(n)$ of encoding bits for $G_n^{(1)}$ and $G_n^{(2)}$. Moreover, we demand $\Pi^{(1)} = \Pi^{(2)} =: \Pi$.

We encode the nodes and distances of $G^{(3)}$ by $\mathcal{N}^{(3)}(n) := \mathcal{N}(n) + 1$ bits. The additional bit $x_{\mathcal{N}(n)}$ of a node number x distinguishes between $G^{(1)}$ and $G^{(2)}$, i. e., $x_{\mathcal{N}(n)} = 1 :\Leftrightarrow v_{|x|} \in V_n^{(1)}$. This new flag bit is inserted at an arbitrary position into the new local variable order $\tau_i \in \Sigma_{\mathcal{N}^{(3)}(n)}$ of each argument $x^{(i)} \in \{0,1\}^{\mathcal{N}^{(3)}(n)}$, $i \in \{1,\ldots,k\}$, yielding the new global order $\Pi^{(3)} := (\pi_n^{(3)})_{n\in\mathbb{N}}$, $\pi_n^{(3)} \in \Sigma_{k\mathcal{N}^{(3)}(n)}$.

Theorem 5 *Let* $G^{(3)}$ *be the cojoin of* $G^{(1)}$ *and* $G^{(2)}$. *If* $C^{(i)}$ *and* $DIST^{(i)}$ *are* b*-bounded by* Π *for* $i \in \{1,2\}$, *then* $C^{(3)}$ *and* $DIST^{(3)}$ *are* $\mathcal{O}(b^2)$*-bounded by* $\Pi^{(3)}$.

Proof. We express $C^{(3)}$ and $DIST^{(3)}$ in terms of $C^{(1)}$, $C^{(2)}$, $DIST^{(1)}$, and $DIST^{(2)}$:

$$C^{(3)}(x,y,d)_n = \big[v_{|x|} \in V_n^{(1)} \wedge v_{|y|} \in V_n^{(1)} \wedge C^{(1)}(x,y,d)_n\big] \vee \big[v_{|x|} \in V_n^{(2)} \wedge v_{|y|} \in V_n^{(2)} \wedge C^{(2)}(x,y,d)_n\big]\ ,$$

$$DIST^{(3)}(x^{(1)},x^{(2)},y^{(1)},y^{(2)},d)_n = \big[v_{|x|} \in V_n^{(1)} \wedge v_{|y|} \in V_n^{(1)} \wedge DIST^{(1)}(x,y,d)_n\big] \vee \big[v_{|x|} \in V_n^{(2)} \wedge v_{|y|} \in V_n^{(2)} \wedge DIST^{(2)}(x,y,d)_n\big]\ .$$

Checks of a node's flag bit $x_{\mathcal{N}(n)}$ are $\mathcal{O}(1)$-bounded by any variable order. Due to Theorem 1, both $C^{(3)}$ and $DIST^{(3)}$ are $\mathcal{O}(b^2)$-bounded by $\Pi^{(3)}$. □

Theorem 6 *Let* $G^{(3)}$ *be the* $\mathcal{A}$*-join of* $G^{(1)}$ *and* $G^{(2)}$. *We assume w. l. o. g. that* $\mathcal{A}(n) < 2^{\mathcal{N}^{(3)}(n)}$ *for all* $n \in \mathbb{N}$. *If* $C^{(i)}$ *and* $DIST^{(i)}$ *are* b*-bounded by* Π *for* $i \in \{1,2\}$, *then* $C^{(3)}$ *and* $DIST^{(3)}$ *are* $\mathcal{O}(b^2)$*-bounded by* $\Pi^{(3)}$.

Proof. We express $C^{(3)}$ and $DIST^{(3)}$ in terms of $C^{(1)}$, $C^{(2)}$, $DIST^{(1)}$, and $DIST^{(2)}$:

$$C^{(3)}(x,y,d)_n = \big[v_{|x|} \in V_n^{(1)} \wedge v_{|y|} \in V_n^{(1)} \wedge C^{(1)}(x,y,d)_n\big] \vee \big[v_{|x|} \in V_n^{(2)} \wedge v_{|y|} \in V_n^{(2)} \wedge C^{(2)}(x,y,d)_n\big] \vee \big[v_{|x|} \in V_n^{(1)} \wedge v_{|y|} \in V_n^{(2)} \wedge \big(|d| = \mathcal{A}(n)\big)\big]\ ,$$

$$\begin{aligned}DIST^{(3)}(x^{(1)},x^{(2)},y^{(1)},y^{(2)},d)_n = &\left[v_{|x|}\in V_n^{(1)} \wedge v_{|y|}\in V_n^{(1)} \wedge DIST^{(1)}(x,y,d)_n\right]\\ &\vee \left[v_{|x|}\in V_n^{(2)} \wedge v_{|y|}\in V_n^{(2)} \wedge DIST^{(2)}(x,y,d)_n\right]\\ &\vee \left[v_{|x|}\in V_n^{(1)} \wedge v_{|y|}\in V_n^{(2)} \wedge \big(|d| = \mathcal{A}(n)\big)\right] .\end{aligned}$$

Checks of a node's flag bit $x_{\mathcal{N}(n)}$ are $\mathcal{O}(1)$-bounded by any variable order. The same holds for the comparison $\big(|d| = \mathcal{A}(n)\big)$. Due to Theorem 1, both $C^{(3)}$ and $DIST^{(3)}$ are $\mathcal{O}(b^2)$-bounded by $\Pi^{(3)}$. □

Remark 2 *For both cojoin and $\mathcal{A}$-join, the resulting function $VALID^{(3)}(x)_n$ can be obtained from $VALID^{(1)}(x)_n$ and $VALID^{(2)}(x)_n$ by*

$$VALID^{(3)}(x)_n := \left[v_{|x|}\in V_n^{(1)} \wedge VALID^{(1)}(x)_n\right] \vee \left[v_{|x|}\in V_n^{(2)} \wedge VALID^{(2)}(x)_n\right] .$$

Analogue to Theorems 5 and 6, a bounded width b of $VALID^{(1)}$ and $VALID^{(2)}$ w. r. t. Π implies $VALID^{(3)}$ to be $\mathcal{O}(b^2)$-bounded by $\Pi^{(3)}$.

4.2 Product Compositions

Now we consider the case $V_n^{(3)} = V_n^{(1)} \times V_n^{(2)}$. Let arguments representing node numbers share the same local variable order $\tau \in \Sigma_{\mathcal{N}^{(i)}(n)}$ within $\pi_n^{(i)}$ for $i \in \{1,2\}$ and all $n \in \mathbb{N}$. Moreover, we demand that bits of distance values are read with increasing significance.

We encode each node number and distance value of $G^{(3)}$ by $\mathcal{N}^{(3)}(n) := \mathcal{N}^{(1)}(n) + \mathcal{N}^{(2)}(n)$ bits. Then, each binary node number $x \in \{0,1\}^{\mathcal{N}^{(3)}(n)}$ consists of two subnumbers $x^{(1)}$ and $x^{(2)}$ and represents the node $(v_{|x^{(1)}|}, v_{|x^{(2)}|}) \in V_n^{(3)}$ (see Sect. 3.2). We use variable orders $\Pi^{(3)}$ respecting both $\Pi^{(1)}$ and $\Pi^{(2)}$ while reading the subnumbers $x^{(1)}$ and $x^{(2)}$ of each node argument 2-interleaved.

Theorem 7 *Let $G^{(3)}$ be the node substitution of $G^{(1)}$ in $G^{(2)}$. If $C^{(i)}$ and $DIST^{(i)}$ are b-bounded by $\Pi^{(i)}$ for $i \in \{1,2\}$, then $C^{(3)}$ is $O(b^2)$-bounded by $\Pi^{(3)}$ and $DIST^{(3)}$ is $2^{\mathcal{O}(b^2)}$-bounded by $\Pi^{(3)}$.*

Proof. We express $C^{(3)}$ in terms of $C^{(1)}$ and $C^{(2)}$:

$$\begin{aligned}C^{(3)}(x^{(1)},x^{(2)},y^{(1)},y^{(2)},d)_n = &\left[C^{(1)}(x^{(1)},y^{(1)},d)_n \wedge (x^{(2)} = y^{(2)})\right]\\ &\vee C^{(2)}(x^{(2)},y^{(2)},d)_n .\end{aligned}$$

Shortest paths between nodes $u := (u^{(1)}, u^{(2)}) \in V^{(3)}$ and $v := (v^{(1)}, v^{(2)}) \in V^{(3)}$ have length $\mathrm{dist}^{(3)}(u,v) = \mathrm{dist}^{(2)}(u^{(2)}, v^{(2)})$ if $u^{(2)} \neq v^{(2)}$. Otherwise, both u and v are in the same copy $u^{(2)} = v^{(2)}$ of $G^{(1)}$. A shortest path $\bar{p}$ may stay in this copy, which implies $\mathrm{dist}^{(3)}(u,v) = \mathrm{dist}^{(1)}(u^{(1)}, v^{(1)})$. Alternatively, $\bar{p}$ may be a detour through other copies of $G^{(1)}$ in $G^{(2)}$. Such a detour corresponds to a cycle

$(u^{(2)}, \ldots, z, \ldots, u^{(2)})$ with $z \neq u^{(2)} = v^{(2)}$. We express and $\mathit{DIST}^{(3)}$ in terms of $\mathit{DIST}^{(1)}$ and $\mathit{DIST}^{(2)}$:

$$\begin{aligned}\mathit{DETOUR}(x^{(2)}, d) := (\exists z, d^{(1)}, d^{(2)})\big[(x^{(2)} \neq z)\\ \wedge \mathit{DIST}^{(2)}(x^{(2)}, z, d^{(1)})_n \wedge \mathit{DIST}^{(2)}(z, x^{(2)}, d^{(2)})_n \wedge (|d^{(1)}| + |d^{(2)}| = |d|)\big] \ ,\end{aligned}$$

$$\begin{aligned}\mathit{DIST}^{(3)}(x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)}, d)_n = \big[(x^{(2)} \neq y^{(2)}) \wedge \mathit{DIST}^{(2)}(x^{(2)}, y^{(2)}, d)_n\big]\\ \vee \Big[(x^{(2)} = y^{(2)}) \wedge \mathit{DIST}^{(1)}(x^{(1)}, y^{(1)}, d)_n\\ \wedge \overline{(\exists d^{(1)})\big[(|d^{(1)}| < |d|) \wedge \mathit{DETOUR}(x^{(2)}, d^{(1)})\big]}\Big]\\ \vee \Big[(x^{(2)} = y^{(2)}) \wedge \mathit{DETOUR}(x^{(2)}, d)\\ \wedge \overline{(\exists d^{(1)})\big[(|d^{(1)}| < |d|) \wedge \mathit{DIST}^{(1)}(x^{(1)}, y^{(1)}, d^{(1)})_n\big]}\Big] \ .\end{aligned}$$

Using the same local variable order τ for all node arguments, comparisons of node numbers like $(x^{(2)} = y^{(2)})$ are $\mathcal{O}(1)$-bounded [19]. The same bound holds for the distance comparison $(|d^{(1)}| + |d^{(2)}| = |d|)$. From Theorem 1 we conclude $C^{(3)}$ to be $O(b^2)$-bounded.

In $\mathit{DIST}^{(3)}$, the quantifier $(\exists d^{(1)})$ for the case of not taking a detour can be gathered with the quantifier $(\exists z, d^{(1)}, d^{(2)})$ in DETOUR. Due to Theorems 1 and 3, the resulting quantification is applied to an intermediate result of width $\mathcal{O}(b^2)$. Due to Theorem 2, this causes an exponentiation leading to the width bound of $2^{\mathcal{O}(b^2)}$. This bound dominates the other bounds of $\mathcal{O}(b)$ and $2^{\mathcal{O}(b)}$. □

Theorem 8 *Let $G^{(3)}$ be the product of $G^{(1)}$ and $G^{(2)}$. If $C^{(i)}$ and $\mathit{DIST}^{(i)}$ are b-bounded by $\Pi^{(i)}$ for $i \in \{1, 2\}$, then $C^{(3)}$ is $O(b^2)$-bounded by $\Pi^{(3)}$ and $\mathit{DIST}^{(3)}$ is $2^{\mathcal{O}(b^2)}$-bounded by $\Pi^{(3)}$.*

Proof. A shortest path in the product $G_n^{(3)}$ is composed of shortest paths in $G_n^{(1)}$ and $G_n^{(2)}$. We express $C^{(3)}$ and $\mathit{DIST}^{(3)}$ in terms of $C^{(1)}$, $C^{(2)}$, $\mathit{DIST}^{(1)}$, and $\mathit{DIST}^{(2)}$:

$$\begin{aligned}C^{(3)}(x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)}, d)_n = \big[C^{(1)}(x^{(1)}, y^{(1)}, d)_n \wedge (x^{(2)} = y^{(2)})\big]\\ \vee \big[(x^{(1)} = y^{(1)}) \wedge C^{(2)}(x^{(2)}, y^{(2)}, d)_n\big] \ ,\end{aligned}$$

$$\begin{aligned}\mathit{DIST}^{(3)}(x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)}, d)_n = (\exists d^{(1)}, d^{(2)}) \ \big[(|d^{(1)}| + |d^{(2)}| = |d|)\\ \wedge \mathit{DIST}^{(1)}(x^{(1)}, y^{(1)}, d^{(1)})_n \wedge \mathit{DIST}^{(2)}(x^{(2)}, y^{(2)}, d^{(2)})_n\big] \ .\end{aligned}$$

Using the same local variable order τ for all node arguments, comparisons of node numbers like $(x^{(1)} = y^{(1)})$ are $\mathcal{O}(1)$-bounded [19]. The same bound holds for the distance comparison ($|d^{(1)}| + |d^{(2)}| = |d|$). From Theorem 1 we conclude $C^{(3)}$ to be $O(b^2)$-bounded.

From Theorems 1 and 3 we conclude that the quantifier in $DIST^{(3)}$ is applied to an intermediate result of width $O(b^2)$. Due to Theorem 2, the quantification causes an exponentiation leading to the final bound of $2^{\mathcal{O}(b^2)}$. □

That is, the efficiency results of Theorem 4 in [18] also hold for complex graphs builded from basic ones having characteristic bounded-width functions.

Remark 3 *For both node substitution and product, the resulting function* $VALID^{(3)}(x^{(1)}, x^{(2)})_n$ *can be obtained from* $VALID^{(1)}(x)_n$ *and* $VALID^{(2)}(x)_n$ *by* $VALID^{(3)}(x^{(1)}, x^{(2)})_n := VALID^{(1)}(x^{(1)})_n \wedge VALID^{(2)}(x^{(2)})_n$. *Due to Theorem 1, a bounded width b of* $VALID^{(1)}$ *and* $VALID^{(2)}$ *w. r. t.* $\Pi^{(1)}$ *resp.* $\Pi^{(2)}$ *implies* $VALID^{(3)}$ *to be* $\mathcal{O}(b^2)$*-bounded by* $\Pi^{(3)}$.

5 Basic Graphs

We now consider sequences of basic undirected graphs without edge weights whose characteristic functions can be composed of multivariate threshold functions. We represent undirected edges as directed symmetric ones. We will make statement on width bounds which can be easily concluded from Theorems 1 and 3.

Let $G := \big(G_n = (V_n, E_n)\big)_{n\in\mathbb{N}}$ be a sequence of directed graphs with $|V_n| = N_n$ and $V_n = \{v_0, \ldots, v_{N-1}\}$. Let $E := \big(E(x, y)_n\big)_{n\in\mathbb{N}}$ be the sequence of G's characteristic functions $E(x, y)_n = 1 :\Leftrightarrow \big(v_{|x|}, v_{|y|}\big) \in E_n$. We use functions $VALID(x)_n$ to represent valid node numbers x with $v_{|x|} \in V_n$. As in Sect. 4, let $\mathcal{N}(n)$ be the number of Boolean variables encoding one node number x and $\Pi := (\pi_n)_{n\in\mathbb{N}}$ be k-interleaved variable orders $\pi_n \in \Sigma_{k\mathcal{N}(n)}$.

Typically, $VALID(x)_n$ would be a comparison with some threshold value having width bound $b = \mathcal{O}(1)$.

Complete Graphs. Complete graphs G_n with $E_n = \{(u, v) \in V^2 \mid u \neq v\}$ have the characteristic function $E(x, y)_n := VALID(x)_n \wedge VALID(y)_n \wedge (x \neq y)$. If x and y have the same the local variable order and $VALID$ is $\mathcal{O}(b)$-bounded by Π, this function is also $\mathcal{O}(b)$-bounded by Π.

Bicliques. Bicliques are complete bipartite graphs. They can be considered as the join of two disjoint complete graphs. The bounded width of their characteristic functions can be concluded from Theorem 6; however, we will explicitly consider the undirected case.

Let $V^{(1)}(x)_n$ and $V^{(2)}(x)_n$ be the characteristic functions of two disjoint subsets $V_n^{(1)}, V_n^{(2)} \subseteq V_n$. Then, the biclique edges are defined by

$$\begin{aligned} E'(x,y)_n &:= V^{(1)}(x)_n \wedge V^{(2)}(y)_n \ , \\ E(x,y)_n &:= E'(x,y)_n \vee E'(y,x)_n \ . \end{aligned}$$

If both $V^{(1)}$ and $V^{(2)}$ are b-bounded by Π, then E is $\mathcal{O}(b^4)$-bounded by Π. As in Sect. 4.1, a flag bit can be used to distinguish between both node subsets. A restriction of the maximum node number of each set can be realized as described in Sect. 3.2. Therefore, E is $\mathcal{O}(1)$-bounded by variable orders Π with increasing bit significance.

Paths. The nodes of a path G_n can be numbered such that $E_n = \big\{(v_a, v_b) \in V_n^2 \mid |a-b| = 1\big\}$. This is expressed symbolically as

$$\begin{aligned} \mathit{VALID}(x)_n &:= (|x| < N_n) \ , \\ E(x,y)_n &:= \mathit{VALID}(x)_n \wedge \mathit{VALID}(y)_n \wedge \big[(|x| - |y| = 1) \vee (|y| - |x| = 1)\big] \ . \end{aligned}$$

If Π reads the node bits with increasing significance, then E is $\mathcal{O}(b^2)$-bounded by Π.

Cycles. The nodes of a cycle G_n can be numbered such that $E_n = \big\{(v_a, v_b) \in V_n^2 \mid |a-b| = 1\big\} \cup \{(v_{N-1}, v_0), (v_0, v_{N-1})\}$. This is expressed symbolically as

$$\begin{aligned} \mathit{VALID}(x)_n &:= (|x| < N_n) \ , \\ E'(x,y)_n &:= (|y| - |x| = 1) \vee \big[(|x| = 0) \wedge (|y| = N_n - 1)\big] \ , \\ E(x,y)_n &:= \mathit{VALID}(x)_n \wedge \mathit{VALID}(y)_n \wedge \big[E'(x,y) \vee E'(y,x)\big] \ . \end{aligned}$$

If Π reads the node bits with increasing significance, then E is $\mathcal{O}(b^2)$-bounded by Π.

Wheels. The nodes of a wheel G_n may be numbered such that

$$\begin{aligned} E'_n = \big\{(v_a, v_b) \in V_n^2 \mid b - a = 1,\ a > 0\big\} \cup \{(v_{N-1}, v_1)\} \\ \cup \big\{(0, v_a) \in V_n^2 \mid a > 0\big\} \ , \\ E_n = \big\{(u,v) \in V_n^2 \mid (u,v) \in E'_n \vee (v,u) \in E'_n\big\} \ . \end{aligned}$$

This is expressed symbolically as

$$\begin{aligned} E'(x,y)_n := (|y| - |x| = 1) \vee \big[(|x| = 1) \wedge (|y| = N_n - 1)\big] \vee \\ \big[(|x| = 0) \wedge (|y| > 0)\big] \ , \end{aligned}$$

$$\begin{aligned} \mathit{VALID}(x)_n &:= (|x| < N_n) \ , \\ E(x,y)_n &:= \mathit{VALID}(x)_n \wedge \mathit{VALID}(y)_n \wedge \big[E'(x,y) \vee E'(y,x)\big] \ . \end{aligned}$$

If Π reads the node bits with increasing significance, then E is $\mathcal{O}(b^2)$-bounded by Π.

Fans. The nodes of a fan G_n can be numbered such that

$$E_n = \big\{(v_a, v_b) \in V^2 \mid |a-b| = 1,\, a, b > 0\big\} \\ \cup \big\{(v_a, v_b) \in V_n^2 \mid (a > 0) \oplus (b > 0)\big\} \ .$$

This is expressed symbolically as

$$VALID(x)_n := (|x| < N_n) \ ,$$

$$E(x,y)_n := VALID(x)_n \wedge VALID(y)_n \\ \wedge \Big[(|x| - |y| = 1) \vee (|y| - |x| = 1) \vee \big[(a > 0) \oplus (b > 0)\big]\Big] \ .$$

If Π reads the node bits with increasing significance, then E is $\mathcal{O}(b^2)$-bounded by Π.

Stars. We consider star graphs G_n with ℓ leaves which are reached by paths of lengths $p_n(1)$ to $p_n(n)$. That is, $N_n = \sum_{i=1}^{\ell} p_n(i) - \ell + 1$. Let be $c_n(i) := \sum_{j=1}^{i-1} p_n(j)$. The nodes can be numbered such that

$$E'_n = \bigcup_{i=1}^{\ell} \Big(\big\{(v_a, v_b) \in V_n^2 \mid b - a = 1,\, a > c_n(i),\, b \le c_n(i+1)\big\} \\ \cup \big\{(v_0, v_a) \mid a = c_n(i) + 1\big\} \Big) \ ,$$

$$E_n = \big\{(u, v) \in V_n^2 \mid (u, v) \in E'_n \vee (v, u) \in E'_n\big\} \ .$$

This is expressed symbolically as

$$E'(x,y)_n := \bigvee_{i=1}^{\ell} \Big[\big[(|y| - |x| = 1) \wedge \big(|x| > c_n(i)\big) \wedge \big(b \le c_n(i+1)\big)\big] \\ \vee \big[(|x| = 0) \wedge \big(|y| = c_n(i) + 1\big)\big]\Big] \ ,$$

$$\begin{aligned} VALID(x)_n &:= (|x| < N_n) \ , \\ E(x,y)_n &:= VALID(x)_n \wedge VALID(y)_n \wedge \big[E'(x,y) \vee E'(y,x)\big] \ . \end{aligned}$$

If Π reads the node bits with increasing significance, then E is $\mathcal{O}(b^2 \cdot a^{2\ell})$-bounded by Π for an appropriate constant a.

Alternatively, we could interprete a binary node numbers x as consisting of two subnumbers $x^{(1)} \in \{0,1\}^{\mathcal{N}^{(1)}(n)}$ and $x^{(2)} \in \{0,1\}^{\mathcal{N}^{(2)}(n)}$ for $x = x^{(1)}x^{(2)}$ and $\mathcal{N}^{(1)}, \mathcal{N}^{(2)} \colon \mathbb{N} \to \mathbb{N}$. As described in Sect. 3.2, we assume that π_n reads $x^{(1)}$ and

$x^{(2)}$ 2-interleaved. Then, the number of leaves may be variable, such that G_n has ℓ_n leaves. Let $|x^{(1)}|$ identify a leave resp. path number, whereas $|x^{(2)}|$ identifies the node position on path $|x^{(1)}|$. We use functions $P(x,d)_n$ with

$$P(x^{(1)},d)_n = 1 :\Leftrightarrow (1 \le |x^{(1)}| \le \ell_n) \wedge \big(p_n(|x^{(1)}| = |d|\big) \ .$$

Path number 0 is reserved for the root node v_0. The star G_n has an edge $(v_{|x|}, v_{|y|})$ iff either both nodes belong to the same path $|x^{(1)}| = |y^{(1)}|$ and $\big||x^{(2)}| - |y^{(2)}|\big| = 1$ or $v_{|x|}$ is the root and $v_{|y|}$ is the first node on a path.

The corresponding definition of $E(x,y)_n$ is

$$E'(x,y)_n := \big[(x^{(1)} = y^{(1)}) \wedge (|y| - |x| = 1)\big] \\ \vee \big[(|x| = 0) \wedge (x \neq y) \wedge (|y^{(2)}| = 0)\big] \ ,$$

$$\begin{aligned} \mathit{VALID}(x)_n &:= (|x| = 0) \vee (\exists d)\ \big[P(x^{(1)},d)_n \wedge (|x^{(2)}| < |d|)\big] \ , \\ E(x,y)_n &:= \mathit{VALID}(x)_n \wedge \mathit{VALID}(y)_n \wedge \big[E'(x,y) \vee E'(y,x)\big] \ . \end{aligned}$$

If P is b-bounded by Π, we conclude E to have bounded width $2^{\mathcal{O}(b)}$ w. r. t. Π. However, there is one disadvantage: If the path lengths $p_n(1), \ldots, p_n(\ell_n)$ are of different magnitudes for each $n \in \mathbb{N}$, the symbolic representation contains many redundant singletons causing $\mathcal{N}(n) = \Omega(\log N_n)$.

Remark 4 *Remember that the negation operation does not change the OBDD-size. That is, if the characteristic functions* $E := (E(x,y)_n)_{n\in\mathbb{N}}$ *are b-bounded by variable orders* Π*, then* $E^{\mathrm{c}} := (\overline{E(x,y)_n})_{n\in\mathbb{N}}$ *is also b-bounded by* Π*.*

If there is a b-bounded node restriction function $\mathit{VALID}(x)_n$*, this can be taken into account by* $E^{\mathrm{c}}(x,y)_n := \mathit{VALID}(x)_n \wedge \mathit{VALID}(y)_n \wedge \overline{E(x,y)_n}$*.*

6 Threshold Graphs

Threshold graphs are a class of undirected graphs which is important to many application areas. They are a subset of cographs. In the following, we denote the degree of a node $v \in V$ by $\deg(v)$. A comprehensive discussion of threshold graphs including many different characterizations can be found in [11].

Theorem 9 (Mahadev et al. [11]) *Consider an undirected graph* $G = (V,E)$*. Let* $\mathcal{D} = \big(d(0), \ldots, d(M)\big) \in \mathbb{N}_0^{M+1}$ *be the degree sequence of* G *with* $i < j \Rightarrow d(i) < d(j)$*. Assume that* $d_0 = 0$*, independent of the existence of singletons.*

The following are equivalent definitions of threshold graphs.

1. *A graph* G *is a threshold graph iff it can be constructed from the empty graph by repeatedly adding either an isolated vertex or a dominating vertex.*

2. A graph $G = (V, E)$ is a threshold graph iff

$$E = \big\{(u, v) \in V^2 \mid d^{-1}\big(\deg(u)\big) + d^{-1}\big(\deg(v)\big) \geq M\big\} \ .$$

We use these two definitions to show that threshold graphs with certain properties have characteristic bounded-width functions.

Therefore, let $G := \big(G_n = (V_n, E_n)\big)_{n\in\mathbb{N}}$ be a sequence of threshold graphs with $|V_n| = N_n$, $V_n = \{v_0, \ldots, v_{N-1}\}$, and degree sequences $\mathcal{D}_n = \big(d_n(0), \ldots, d_n(M_n)\big)$. Assume that $d_n(0) = 0$ for all $n \in \mathbb{N}$. Let $E := \big(E(x, y)_n\big)_{n\in\mathbb{N}}$ be the sequence of G's characteristic functions $E(x, y)_n = 1 :\Leftrightarrow (v_{|x|}, v_{|y|}) \in E_n$. Let $\mathcal{N}(n)$ be the number of Boolean variables encoding one node number x and $\Pi := (\pi_n)_{n\in\mathbb{N}}$ be k-interleaved variable orders $\pi_n \in \Sigma_{k\mathcal{N}(n)}$ with increasing bit significance.

For all following kinds of symbolic representations, the valid node number are defined by $\mathit{VALID}(x)_n := (|x| < N_n)$.

6.1 Pleasant Composition Sequences

We present different conditions connected to the construction sequence of each G_n which imply E to have bounded width w. r. t. Π.

Numbering the nodes by construction order. Assume that the nodes of G_n are numbered by the order in which they were added during the successive construction of G_n for all $n \in \mathbb{N}$. That is, v_a was the ath vertex added to G_n. It is reasonable to choose $\mathcal{N}(n) := \lceil \log N_n \rceil$. Consider the characteristic function $\mathit{KIND}(x)_n$ which reflects the construction sequence:

$$\mathit{KIND}(x)_n = 1 :\Leftrightarrow \text{Node } v_{|x|} \text{ was added dominating.}$$

Obviously, a vertex v_a is adjacent to v_b for $a < b$ iff v_b was added dominating. Hence, it is

$$\begin{aligned} E(x, y)_n = {} & (|x| < N_n) \wedge (|y| < N_n) \\ & \wedge \big[(|x| < |y|) \wedge \mathit{KIND}(y)\big] \vee \big[(|x| > |y|) \wedge \mathit{KIND}(x)\big] \ . \end{aligned}$$

We conclude that if KIND has bounded width b w. r. t. Π, E is $\mathcal{O}(b^2)$-bounded by Π.

Numbering the nodes by type of adjacency. Assume that N_n^{d} nodes have been added dominating to G_n, whereas $N_n^{\mathrm{i}} = N_n - N_n^{\mathrm{d}}$ have been added isolated G_n for all $n \in \mathbb{N}$. Moreover, assume that the nodes $v_0, \ldots, v_{N^{\mathrm{d}_n}-1}$ have been added dominating, while the nodes $v_{N^{\mathrm{d}_n}}, \ldots, v_{N_n-1}$ have been added isolated.

Again, we choose $\mathcal{N}(n) := \lceil \log N_n \rceil$. This time, we consider the characteristic function $ORDER(x,y)_n$ with

$$ORDER(x,y)_n = 1 :\Leftrightarrow \text{Node } v_{|x|} \text{ was added before node } v_{|y|} \text{ to } G_n.$$

Obviously, a dominating vertex v_a is adjacent to v_b if it was added after v_b. Hence, it is

$$\begin{aligned} E(x,y)_n = (|x| < N_n) \wedge (|y| < N_n) \\ \wedge \left[(|x| < N_n^{\mathrm{d}}) \wedge ORDER(y,x)_n\right] \vee \left[(|y| < N_n^{\mathrm{d}}) \wedge ORDER(x,y)_n\right] \ . \end{aligned}$$

We conclude that if $ORDER$ has bounded width b w. r. t. Π, E is $\mathcal{O}(b^2)$-bounded by Π.

6.2 Pleasant Degree Sequences

At next, we present different conditions connected to the degree sequence of each G_n which imply E to have bounded width w. r. t. Π.

Positions within the degree sequence. Consider the characteristic function $DEGPOS(x,d)_n$ defined by

$$DEGPOS(x,y)_n = 1 :\Leftrightarrow d_n^{-1}\big(\deg(v_{|x|})\big) = |y| \ .$$

The second characterization of threshold graphs in Theorem 9 leads to the following symbolic formulation of $E(x,y)_n$:

$$\begin{aligned} E(x,y)_n := (|x| < N_n) \wedge (|y| < N_n) \wedge (\exists z^{(1)}, z^{(2)}) \left[(z^{(1)} + z^{(2)} \geq M_n)\right. \\ \left.\wedge\, DEGPOS(x, z^{(1)})_n \wedge DEGPOS(y, z^{(2)})_n\right] \ . \end{aligned}$$

We conclude that if $DEGPOS$ has bounded width b w. r. t. Π, E is $\mathcal{O}(b^2)$-bounded by Π.

Distribution of degrees. Consider the characteristic function $DEGSIZE(x,y)_n$ defined by

$$\begin{aligned} DEGSIZE(x,y)_n = 1 :\Leftrightarrow (|x| \leq M_n) \\ \wedge \left|\left\{v_{|x|} \in V_n \mid d_n^{-1}\big(\deg(v_{|x|})\big) = |x|\right\}\right| = |y| \ . \end{aligned}$$

That is, $DEGSIZE(x,y)$ represents pairs (x,y) such that the number of nodes whose degree has position $|x|$ in $\mathcal{D}_n$ is $|y|$.

We propose the following node numbering scheme: Assume that node numbers x consist of two subnumbers $x^{(1)} \in \{0,1\}^{\mathcal{N}^{(1)}(n)}$ and $x^{(2)} \in \{0,1\}^{\mathcal{N}^{(2)}(n)}$ for $x =$

$x^{(1)}x^{(2)}$ and $\mathcal{N}^{(1)}, \mathcal{N}^{(2)}\colon \mathbb{N} \to \mathbb{N}$. As described in Sect. 3.2, we assume that π_n reads $x^{(1)}$ and $x^{(2)}$ 2-interleaved. Let $x^{(1)} = d_n^{-1}\big(\deg(v_{|x|})\big)$.

The second characterization of threshold graphs in Theorem 9 leads to the following symbolic formulation of $E(x,y)_n$:

$$E(x,y)_n := (|x^{(1)}| + |y^{(2)}| \geq M_n) \\ \wedge (\exists z)\ \big[(|x^{(2)}| < |z|) \wedge \mathit{DEGSIZE}(x^{(1)}, z)\big] \\ \wedge (\exists z)\ \big[(|y^{(2)}| < |z|) \wedge \mathit{DEGSIZE}(y^{(1)}, z)\big] \ .$$

We conclude that if *DEGSIZE* has bounded width b w.r.t. Π, E is $2^{\mathcal{O}(b)}$-bounded by Π.

This node numbering scheme may violate our assumption $\mathcal{N}(n) = \lceil \log N_n \rceil$. Moreover, if the number $\big|\{v \in V_n \mid d_n^{-1}\big(\deg(v)\big) = i\}\big|$ of nodes having degree i variates strongly for different i, it may be $\mathcal{N}(n) = \Omega(\log N_n)$.

7 Symbolic APSP-Algorithms on Graphs with Characteristic Bounded-Width Functions

In [18], a symbolic algorithm for the all-pairs shortest-paths (APSP) problem is presented. It works on symbolically represented loopless directed graphs $G = (V, E, c)$ with strictly positive integral edge weights $c\colon E \to \mathbb{N}_{>0}$.

The maximum path length in G is $B(N-1) =: L$. Let $n := \lceil \log(L+1) \rceil = \Theta(\log N + \log B)$ the number of bits encoding one node number or distance value. The algorithm receives the input graph G as an OBDD for the characteristic functions $C(x, y, d)$ with

$$C(x,y,d) = 1 :\Leftrightarrow \big[(v_{|x|}, v_{|y|}) \in E\big] \wedge \big[c(v_{|x|}, v_{|y|}) = |d|\big] \ .$$

Let $\mathrm{dist}\colon V^2 \to \mathbb{N}_0 \cup \{\infty\}$ be the shortest-paths function mapping node pairs (u, v) to the length of shortest u–v-paths. The algorithm's primary output is the OBDD for dist's characteristic function $\mathit{DIST}(x, y, d)$ defined by

$$\mathit{DIST}(x,y,d) = 1 :\Leftrightarrow \mathrm{dist}(v_{|x|}, v_{|y|}) = |d| \ .$$

In order to compute $\mathit{DIST}(x, y, d)$, it computes intermediate results $D_i(x, y, d) = \mathit{DIST}(x, y, d) \wedge (|d| < 2^i)$ for $i \in \{0, \ldots, n\}$ by executing a number of $\Theta\big(\log^2(NB)\big)$ OBDD-operations.

We consider graph sequences $G := (G_n)_{n\in\mathbb{N}}$ with same notation as in Sect. 4 and additionally demand Π to read distance values with increasing significance.

Theorem 10 (Sawitzki [18]) *If C and DIST are b-bounded by Π, the symbolic APSP-algorithm computes $\mathit{DIST}(x,y,d)_n$ from $C(x,y,d)_n$ in time and space $\mathcal{O}\big(\log(N_n B_N) \cdot 2^{\mathcal{O}(b^3)}\big)$ for all $n \in \mathbb{N}$.*

The proof of this analysis result makes use of the fact that the intermediate functions D_i are $\mathcal{O}(b)$-bounded by Π. This in turn holds because the algorithms iterates over the length $|d|$ of considered shortest paths.

Instead, we could iterate over the number of edges of considered paths. This approach is able to handle general integral edge weights. In the following, we call this the edge-iteration-algorithm (EI-algorithm). However, we will show that not all of its intermediate functions have bounded-width if input and output do so.

The EI-algorithm produces intermediate functions $D_i'(x,y,d)_n$ defined by

$$D_i'(x,y,d)_n = 1 :\Leftrightarrow \left\{ \begin{array}{c} \text{There is a } v_{|x|}\text{–}v_{|y|}\text{-path} \\ \bar{p} = (v_{|x|} = p_0, \ldots, p_K = v_{|y|}) \text{ in } G_n \text{ with} \\ \|\bar{p}\| = \operatorname{dist}(v_{|x|}, v_{|y|}) = |d| \text{ and } K \leq 2^i. \end{array} \right.$$

There seems to be no direct restriction of $DIST(x,y,d)$ to $D_i'(x,y,d)$ by a simple comparison like $(|d| < 2^i)$. Nevertheless, a bounded width of C and $DIST$ still could imply bounded width of D_i' for all $i \in \mathbb{N}$. We show that this is not the case by constructing a counterexample graph sequence G^*.

7.1 Definition of G^*

We construct a graph sequence $G^* := \big(G_n^* = (V_n^*, E_n^*, c_n^*)\big)_{n\in\mathbb{N}_{>0}}$ such that both its characteristic functions C^* as well as $DIST^*$ have bounded width w. r. t. k-interleaved variable orders $\Pi^* := (\pi_n^*)_{n\in\mathbb{N}}$, $\pi_n^* \in \Sigma_{k\mathcal{N}^*(n)}$. The number $\mathcal{N}^*(n)$ of Boolean variables encoding one binary node number or distance value will be of magnitude $\Theta\big(\log(V_n^* B_n^*)\big)$ for $B_n^* := \max\{c_n^*(e) \mid e \in E_n^*\}$. Hence, it fulfills the conditions of Theorem 10, and the symbolic APSP-algorithm presented in [18] computes $DIST_n^*$ in polylogarithmic time and space w. r. t. $N_n^* := |V_n^*|$ and B_n^* for all $n \in \mathbb{N}$.

The graph G_n^* consists of subgraphs $G_n^{(2^{n-1}+1)}, \ldots, G_n^{(2^n-1)}$ that share only one special node $s \in V_n^*$. Subgraph $G_n^{(i)}$ has the node set $V_n^{(i)}$ defined by

$$\begin{aligned} U &:= \{u_0, \ldots, u_{2^n-1}\} \;, \\ W &:= \{w_{2^n-i}, \ldots, w_{2^n-1}\} \;, \\ V_n^{(i)} &:= \{s\} \cup U \cup W \;. \end{aligned}$$

Moreover, the edge set $E_n^{(i)}$ of $G_n^{(i)}$ is defined by

$$\begin{aligned} a &:= (u_{2^n-i-1}, u_{2^n-i}) \;, \\ A &:= \big\{(u_j, u_{j+1}) \mid 2^n - i \leq j \leq 2^n - 2\big\} \;, \\ B_1 &:= \big\{(s, u_0)\big\} \cup \big\{(u_j, u_{j+1}) \mid 0 \leq j \leq 2^n - i - 2\big\} \;, \\ B_2 &:= \big\{(u_j, w_{j+1}),\ (w_{j+1}, u_{j+1}) \mid 2^n - i - 1 \leq j \leq 2^n - 2\big\} \;, \\ B &:= B_1 \cup B_2 \;, \\ E_n^{(i)} &:= \{a\} \cup A \cup B \end{aligned}$$

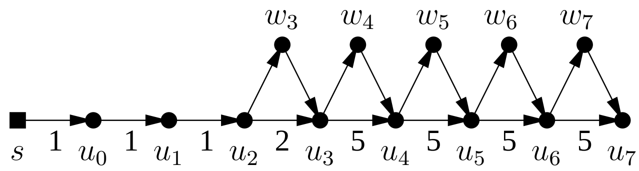

Figure 1: The subgraph $G_3^{(5)}$. Edges incident to w-nodes have weight 1. The path $(s, u_0, \ldots, u_7)$ has length $5^2 = 25$. The shortest s–u_7-path uses detours via w-nodes and has length $2^3 + 5 = 13$.

and is weighted by $c_n^{(i)}$ with

$$c_n^{(i)}(e) := \begin{cases} 2i - 2^n & , e = a \\ i & , e \in A \\ 1 & , e \in B \end{cases} .$$

That is, every subgraph $G_n^{(i)}$ contains a path $(s, u_0, \ldots, u_{2^n-1})$ consisting of 2^n edges. Its first $2^n - i$ edges have weight 1, edge a has weight $2i - 2^n$, and the remaining $i - 1$ edges have weight i. We conclude that the over-all weight of this path is $(2^n - i) + (2i - 2^n) + (i - 1)i = i^2$. Furthermore, every edge from the set $a \cup A$ is bridged by two edges from B_2. Note that all edge weights are strictly positive. Figure 1 sketches the layout of $G_3^{(5)}$.

Symbolic Representation of G^*. Being trained in expressing characteristic functions in terms of multivariate threshold functions (see Sects. 5 and 6), we clearly see that $C^*(x, y, d)_n$ can be composed of multivariate threshold functions with constant maximum absolute weight. Therefore, a k-interleaved variable order π_n^* with increasing bit significance is used.

Each node number x is interpreted as consisting of two subnumbers $x^{(1)} \in \{0,1\}^{\mathcal{N}^*(n)-n-1}$ and $x^{(2)} \in \{0,1\}^{n+1}$, the former identifying the subgraph number i, the latter identifying the concrete node within $G_n^{(i)}$. The most significant bit $x_n^{(2)}$ of $x^{(2)}$ flags if x belongs to node set U ($x_n^{(2)} = 1$) or W ($x_n^{(2)} = 0$) of $G^{(|x^{(1)}|)}$. The bits $x_{n-1}^{(2)}, \ldots, x_0^{(2)}$ identify the concrete node number within the set U resp. W. Subgraph number $i = 0$ may be reserved for the shared starting node $s =: v_0$.

Due to $2^n < N_n^* = 1 + \sum_{i=2^{n-1}+1}^{2^n-1} (2^n + i - 1) < 2^{3n}$ and $B_n^* = 2^n - 1$, we choose $\mathcal{N}^*(n) := 3n + 1 = \Theta\big(\log(N_n^* B_n^*)\big)$.

Corollary 1 *The characteristic functions C^* for G^* are $\mathcal{O}(1)$-bounded by Π.*

7.2 Shortest Paths in G^*

We now consider the lengths of all pairs' shortest paths in $G_n^{(i)}$. The nodes have increasing indices on all paths. All edges except those of $\{a\} \cup A =: H$ have weight 1. Each edge of H can be bridged by two edges of B_2. We conclude

$$\begin{aligned}
\mathrm{dist}(u_j, u_{j'}) &= \max\big\{(2^n - i - 1) - j, 0\big\} - \max\big\{(2^n - i - 1) - j', 0\big\} \\
&\quad +2 \cdot \Big[\max\big\{j' - (2^n - i - 1), 0\big\} - \max\big\{j - (2^n - i - 1), 0\big\}\Big] \\
&\leq 2 \cdot (j' - j) \ , \\
\mathrm{dist}(u_j, w_{j'}) &= \mathrm{dist}(u_j, u_{j'-1}) + 1 \ , \\
\mathrm{dist}(w_j, u_{j'}) &= \mathrm{dist}(u_j, u_{j'}) + 1 \ , \\
\mathrm{dist}(w_j, w_{j'}) &= \mathrm{dist}(u_j, u_{j'-1}) + 2 \ ,
\end{aligned}$$

and $\mathrm{dist}(s, v) = \mathrm{dist}(u_0, v) + 1$ for all $0 \leq j \leq j' \leq 2^n - 1$ and $v \in V_n^{(i)} \setminus \{s\}$.

Again, it is easy to see that $DIST^*(x, y, d)_n$ can be composed of multivariate threshold functions covering the distinct combinations of start- and end-nodes.

Corollary 2 *The characteristic functions $DIST^*$ for G^* are $\mathcal{O}(1)$-bounded by Π.*

7.3 Shortest Paths in G^* Consisting of at Most 2^n Edges

The shortest path from s to $u_{2^n-1} \in V_n^{(i)}$ consists of $\mathrm{dist}(s, u_{2^n-1}) = 2^n + i$ edges each of weight 1. We consider the intermediate function $D_n'^*(x, y, d)_n$ computed by the EI-algorithm: Its definition allows only paths consisting of at most 2^n edges. That is, the shortest s–u_{2^n-1}-paths are not covered by $D_n'^*(x, y, d)_n$. Instead, it has to represent the direct paths of length $i^2 = |y^{(1)}|^2$ using all 2^n edges from $H \cup B_1$. Each detour via an edge of B_2 would increase the number of visited edges by 1.

Theorem 11 *There is no constant $b \in \mathbb{N}$ and variable order sequence $\Pi := (\pi_n)_{n\in\mathbb{N}}$, $\pi_n \in \Sigma_{k\mathcal{N}(n)}$, such that $D_n'^*$ is b-bounded by Π.*

Proof. Assume that $D_n'^*$ is b-bounded by Π. By replacing argument x of $D_n'^*(x, y, d)_n$ by $0^{\mathcal{N}^*(n)}$ (i. e., the start node s) and argument $y^{(2)}$ by 1^{n+1} (i. e., the node u_{2^n-1} of $G_n^{(|y^{(1)}|)}$), we obtain the function sequence $S_1 := \big(S_1(y^{(1)}, d)_n\big)_{n\in\mathbb{N}}$ with $S_1(y^{(1)}, d)_n = 1 \Leftrightarrow (2^{n-1} < |y^{(1)}| < 2^n) \wedge (|y^{(1)}|^2 = |d|)$. We continue with replacing variable $y_{n-1}^{(1)}$ by 1 and obtain $S_2 := \big(S_2(y^{(1)}, d)_n\big)_{n\in\mathbb{N}}$ with $S_2(y^{(1)}, d)_n = 1 \Leftrightarrow (0 < |y^{(1)}| < 2^{n-1}) \wedge \big((|y^{(1)}| + 2^{n-1})^2 = |d|\big)$. In order to cover the case $|y^{(1)}| = 0$, we compute $S_3 := \big(S_3(y^{(1)}, d)_n\big)_{n\in\mathbb{N}}$ with $S_3(y^{(1)}, d)_n := S_2(y^{(1)}, d)_n \vee (|y^{(1)}| = |d| = 0)$. Comparisons with 0 have width 1 for every variable order. Due to Theorem 1, S_3 is $\mathcal{O}(b)$-bounded by Π.

Consider the *squaring function* $SQU_{n,i} \in B_n$ defined on variables $z \in \{0,1\}^n$ which maps the n-bit number z to the ith bit $(|z|^2)_i$ of $|z|^2$, that is, $SQU_{n,i}(z) := (|z|^2)_i$. It is known that there is an $i_n^* \in \{0,\ldots,2n-1\}$ such that $SQU_{n,i^*} \in B_n$ has exponential π-OBDD-size for every variable order $\pi \in \Sigma_n$ (see [19]). We compute $S_4 := \left(S_4(y^{(1)}, d_{i_n^*})_n\right)_{n\in\mathbb{N}}$ with $S_4(y^{(1)}, d_{i_n^*})_n := (\exists d_{\mathcal{N}^*(n)-1},\ldots,d_{i^*+1},d_{i^*-1},\ldots,d_0)\ S_3(y^{(1)},d)_n$. Finally, we obtain $S_5 := \left(S_5(y^{(1)})_n\right)_{n\in\mathbb{N}}$ by replacing variable d_{i^*} by 1. Due to Theorem 2, S_5 is $2^{\mathcal{O}(b)}$-bounded by Π.

We make a read once projection from SQU_{n,i^*} to $S_5(y^{(1)})_{2n}$ by replacing $y^{(1)}_{\mathcal{N}^*(2n)-n-2}\ldots y^{(1)}_n$ by $0^{\mathcal{N}^*(2n)-3n-1}10^{n-1}$ and projecting z onto $y^{(1)}_{n-1}\ldots y^{(1)}_0$. This is legitimate because the i^*th bit of $|z|^2$ equals the i^*th bit of $(|z|+2^{2n-1})^2$. The bounded width of S_5 contradicts the exponential lower bound for SQU_{n,i^*}. □

We have proved that the EI-algorithm does not guaranty all intermediate results D_i' to have bounded width if input C and output $DIST$ do so. There is no symbolic all-pairs shortest-paths algorithm known to the author which has this property and is able to handle general integral edge weights, too.

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