Optimizing misdirection

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Abstract

In this paper we consider the following problem. Given a $(d+1)$-claw free graph $G = (V, E, w)$ where $w : V \to \mathbf{R}_+$, maximize $w(A)$ where A is an independent set in G. Our focus is to minimize the approximation ratio (optimum/obtained) in polynomial time that does not depend on d. Our approach is to apply local improvements of size 2, using a "misdirected" criterion, *i.e.* $w^\alpha(A)$ rather than $w(A)$. We find the optimal value of α for every d, and the resulting ratio is roughly $0.667d$ for $d = 3$, $0.651d$ for $d = 4$ and $0.646d$ for $d > 4$.

1 Introduction

Given a graph $G = (V, E)$, a set $A \subset V$ is *independent* if no edge from E is contained in A. Given $w : V \to \mathbf{R}_+$, the maximum weight independent set problem, w-MIS for short, is to maximize $w(A)$ where A is an independent set. Håstad [10] proved that MIS problem (w-MIS where w is a constant function) is hard to approximate within $|V|^{1-\epsilon}$ ratio. This suggests that we should investigate approximation algorithms for restricted classes of graphs.

One natural restriction is to limit the *degree* of the graph, *i.e.* the number of neighbors that a node may have. A more liberal restriction is to limit the number of *independent* neighbors of a node by some number d. This property can be expressed in terms of a forbidden induced subgraph called $(d+1)$-*claw*, which consists of an independent set of $d+1$ nodes (*talons*) and the *center* node that is connected to all the talons. A graph without induced $(d+1)$-claws is called $(d+1)$-claw free. Such graphs arise naturally in many applications. An application of w-MIS in 5-claw free graphs in a problem from computational biology is given by Bafna *et al.* [4]. Another motivation for studying w-MIS in $(d+1)$-claw free graphs is that it expresses the set packing problem [12]: given a collection of sets that have at most d elements, and a profit associated with each set, find a pairwise disjoint sub-collection with the maximum profit. These problems find also applications to combinatorial auctions [7, 14].

Previous results. Hochbaum [11] has given a $\Delta/2$-approximation[1] for w-MIS on graphs with degree Δ. Halldórsson and Lau [9] improved that to $(\Delta+2)/3$-approximation. For $\Delta = 3$, MIS problem is MAX-SNP-hard [1]. Alon *et al.* [2] proved that MIS is hard to approximate within $O(\Delta^\epsilon)$ for some $\epsilon > 0$.

Now, we consider w-MIS on $(d+1)$-claw free graphs. When $d = 2$, the problem can be solved exactly in polynomial time – see the paper by Nakamura and Tamura [13]. For $d > 2$, a d-approximation with a running time $O(|V|^2)$ was described by Hochbaum [11], and a $|V|^{O(1/\epsilon)}$-time $(d-1+\epsilon)$-approximation for any fixed $\epsilon > 0$, by Bafna *et al.* [4] and Arkin & Hassin [3]. This is the best known to date approximation with running time independent from d. Chandra and Halldórsson [6] give a $2/3\,(d+1)$-approximation for this problem, but the running time is $\Omega(|V|^d)$ and thus it is not polynomial if d is not a fixed constant. Berman [5] improved this to $1/2\,(d+1)$-approximation, but his running time is also $\Omega(|V|^d)$.

Previous results and techniques. The $(d+1)$-claw free graphs form the broadest natural family of graphs where algorithms for MIS have constant approximation ratio, when d is constant. We always assume that $G = (V, E)$ is the input graph. Let $A, B \subseteq V$ and $N(A, B) = \{u \in B : \exists v \in A \text{ such that } \{u, v\} \in E \text{ or } u = v\}$. Even the following simple algorithm assures a ratio of d:

```
GREEDY
  A ← ∅
  while V − N(A,V) ≠ ∅
     u ← an element of V−N(A,V)
     A ← A ∪ {u}
```

The above GREEDY algorithm achieves the same approximation ratio for w-MIS, once we change the greedy selection as follows:

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[1] An α-approximation for a maximization problem Π is a polynomial time algorithm that for every instance of Π finds a feasible solution with a profit at most α times smaller than the optimum; α is called an approximation ratio.

$u \leftarrow$ an element of $V - N(A,V)$ with maximum $w(u)$.

An obvious challenge is to find polynomial time algorithms with better approximation. Given $A, C \subseteq V$, define $A \triangleleft C = (A - N(C,A)) \cup C$; observe that if A and C are independent, so is $A \triangleleft C$. We also say that C *improves* $f(A)$ iff C is independent and $f(A \triangleleft C) > f(A)$. Algorithm 2-IMP below is a $\frac{1}{2}(d+1)$-approximation for MIS on a $(d+1)$-claw free graph [4].

```
2-IMP
  A ← ∅
  while some {u, v} improves |A|
    A ← A ◁ {u, v}
```

By considering improvements with sets of larger sizes, one obtains polynomial time algorithms with ratios approaching $\frac{1}{2}d$ [12]. But, it was not obvious how to extend this idea to w-MIS in d-claw free graphs. We can modify 2-IMP by replacing "improves $|A|$" with "improves $w(A)$," but then, the approximation ratio increases to $d - \frac{1}{2}$. A tight example is a quadrangle free d-regular bipartite graph in which the nodes of the optimum solution form one part, and have weight $2d-1$, while the nodes selected by the algorithm form another part and have weight 2.

Chandra and Halldórsson [6] have found an algorithm that has ratio $\frac{2}{3}(d+1)$. This algorithm stops when no set of talons of a claw improves $w(A)$. While its running time is not polynomial, by an appropriate rounding of the weights we can limit the number of improvements to $O(nk)$ if we multiply the approximation ratio with $1 + \frac{1}{k}$. The searching space implicit in **while** can have size $\Omega(n^d)$. As a result, the running time is $\Omega(n^d)$.

Berman [5] decreased the approximation ratio to $\frac{1}{2}(d+1)$ by changing the condition for improving the candidate set: improve $w^2(A)$ rather than $w(A)$. This approach can be called "misdirected local search": we use "wrong" objective function to guide the search; while individual steps in the search can decrease the objective function, it is much more difficult to get stuck with a very inferior solution.

Our contributions. Chandra and Halldórsson left as an open problem to find an algorithm for w-MIS with ratio at most $\frac{2}{3}(d+1)$, like 2-IMP for w-MIS, with the running time that does not depend on d. In this paper we solve this problem affirmatively. Our algorithm is very similar to 2-IMP but is modified by the use of misdirection, *i.e.* the use of a "wrong" objective function $w^\alpha(A)$. We establish the optimal values of α and the resulting approximation ratios which do not exceed $\frac{2}{3}d$. Somewhat surprisingly, one value of α is optimal for $d = 3$, another for $d = 4$ and the third is optimal for every $d > 4$. Our proofs involve some calculus on two-dimensional functions and, for $d > 3$, they are assisted by a very simple computer program.

2 Algorithm 2-IMP$^\alpha$

Given $X \in V$, define $w^\alpha(X) = \sum_{x \in X}(w(x))^\alpha$. Our algorithm is the following variation of 2-IMP:

```
2-IMP^α
  A ← ∅
  while some {u, v} improves w^α(A)
    A ← A ◁ {u, v}
```

The exponent α which we will establish later satisfies $1 < \alpha < 2$. For each integer k, we can modify algorithm 2-IMP$^\alpha$ to run in polynomial time using the same method as in [5, 6]. First find an approximate solution A using algorithm GREEDY. Then, rescale the solution so that $w(A) = k \cdot |V| = kn$, and run algorithm 2-IMP$^\alpha$ after replacing the function w with $\overline{w}(u) = \lfloor w(u)^\alpha \rfloor^{1/\alpha}$. The function $\overline{w}$ has the following properties: (1) $\overline{w}(u)^\alpha$ is an integer, (2) $0 \le w(u) - \overline{w}(u) < 1$.

Every iteration of the modified 2-IMP$^\alpha$ increases $\overline{w}(A)^\alpha$, and because this value is always integer, the number of iterations is bounded by the maximum value of $\overline{w}^\alpha(A') \le (\overline{w}(A'))^\alpha \le (w(A'))^\alpha \le (dkn)^\alpha$. Because we can implement an iteration in time $O(d^2n^2)$, the modified 2-IMP$^\alpha$ runs in time $O(k^\alpha (dn)^{2+\alpha})$.

Assume that 2-IMP$^\alpha$ is an x-approximation and that A^* is an independent set that maximizes $w(A^*)$. Then the modified 2-IMP$^\alpha$ returns A' that satisfies

$$w(A') \ge \overline{w}(A') \ge x^{-1}\overline{w}(A^*) > x^{-1}(w(A^*) - |A^*|) =$$

$$\left(1 - \frac{|A^*|}{w(A^*)}\right) x^{-1} w(A^*) > \left(1 - \frac{n}{kn}\right) x^{-1} w(A^*),$$

and thus the modified 2-IMP$^\alpha$ is an $\frac{k}{k-1}x$-approximation.

It was already established in [5] that it is helpful to use improvements that improve $w^\alpha(A)$ rather than $w(A)$. In that paper, $\alpha = 2$ was used to obtain the approximation ratio $\frac{1}{2}(d+1)$ using the same family of improvement sets as Chandra and Halldórsson. While matching the ratio of 2-IMP algorithm for MIS, this algorithm was even slower than the one in [6].

In this paper we establish the optimum values of α for all possible values of d. First, let us observe that α should not be too high.

LEMMA 2.1. *The approximation ratio of* 2-IMP$^\alpha$ *is at least* $d(\frac{1}{2})^{1/\alpha}$.

Proof. Let $\beta = (\frac{1}{2})^{1/\alpha}$. Consider a graph that is a d-claw. The center has weight 1 and the talons have weight β. Our algorithm selects the center. If we consider two talons, u and v, then $w^\alpha(u) = w^\alpha(v) =$

$1/2$. Because replacing the center with a pair of talons does not increase $w^\alpha(A)$, the algorithm terminates. ❑

(The notation $\beta = (1/2)^{1/\alpha}$ will be used *ad nauseam* throughout the paper. Note that β is an increasing function of α.) We can also observe that α should not be too low.

LEMMA 2.2. *The approximation ratio of* 2-IMP$^\alpha$ *is at least*

$$1 + \frac{d^2 - 2d + 1}{d - 1 + d\beta}.$$

Proof. Consider a graph with the node set $A \cup B \cup C \cup D$. Our solution is $A \cup B$, where A consists of d nodes of weight β and B consists of $d-1$ nodes of weight 1. The optimum solution is $C \cup D$ where C is a set of $d(d-1)$ nodes of weight 1 and D is a set of d nodes of weight β. Nodes of A and D are connected with a matching. For every $a \in A$ and $b \in B$ there is exactly one $c \in C$ that is connected to both a and b (and thus $|C| = |A| \times |B|$). It is easy to verify that no two elements of $C \cup D$ can improve $w^\alpha(A \cup B)$. We finish the proof by computing the ratio of $w(C \cup D) = d^2 - d + d\beta$ and $w(A \cup B) = d - 1 + d\beta$. ❑

LEMMA 2.3. *For $d \geq 3$ the approximation ratio of* 2-IMP$^\alpha$ *is at least*

$$\frac{\sqrt{5d^2 - 8d + 4} + 2 - d}{2}.$$

Proof. Consider β_0 such that

$$L = d\beta_0 = 1 + \frac{d^2 - 2d + 1}{d - 1 + d\beta_0} = R.$$

By Lemma 2.1, if $\beta \geq \beta_0$ then the approximation ratio of 2-IMP$^\alpha$ is it least L. By Lemma 2.2, if $\beta \leq \beta_0$, then the approximation ratio of 2-IMP$^\alpha$ is at least R. Thus the ratio cannot be smaller than $d\beta_0$. If we replace $d\beta_0$ with β, we can compute it as the positive solution of $\beta = 1 + (d^2 - 2d + 1)/(d - 1 + \beta)$, or, equivalently, $\beta^2 + (d-2)\beta - (d^2 - d) = 0$. ❑

LEMMA 2.4. *For $d \geq 3$, the approximation ratio of* 2-IMP$^\alpha$ *is at least* $d(1/2)^{1/\log_2 3}$.

Proof. Let $2\beta' = (3/2)^{1/\alpha}$. We can construct the following example: the solution of 2-IMP$^\alpha$ is a set A of $d+1$ nodes of weight 1, and the optimum solution is a set B of $(d+1)d/2$ nodes of weight $2\beta'$; for every pair of nodes from A there is exactly one node in B that is connected to both of them. If we try to insert two nodes of B, we have to remove at least three nodes of A, and it is easy to see that this would not improve $w^\alpha(A)$, as $(2\beta')^\alpha = 3$. Clearly, $w(B)/w(A) = d\beta'$.

Note that the lower bound $d\beta'$ is a decreasing function of α, while the lower bound $d\beta$ implied by lemma 2.1 is an increasing function. Suppose that for α_0 we have $\beta = \beta'$; then that estimate is valid for all α's. To find such α and β we solve the equation

$$\beta = (1/2)^{1/\alpha} = (3/2)^{1/\alpha}/2 = \beta' \equiv 2 = 3^{1/\alpha} \equiv$$
$$2^\alpha = 3 \equiv \alpha = \log_2 3.$$

❑

One can check that for $d \leq 4$ the estimate from Lemma 2.3 is larger than the estimate from Lemma 2.4, and that for $d \geq 5$ the converse is true. In the remainder of the paper we will show that these estimates in fact form the exact characterizations, thus proving the following theorem:

THEOREM 2.1. *Consider $d \geq 3$ and the weighted MIS problem on a $(d+1)$-claw free graph $G = (V, E)$. There exists $\beta_d > 0$ and $\alpha_d > 0$ such that $\beta_d^{\alpha_d} = 1/2$, and such that*

- *for $\alpha = \alpha_d$ algorithm* 2-IMP$^\alpha$ *has approximation ratio $\beta_d d$;*
- *for $\alpha \neq \alpha_d$ algorithm* 2-IMP$^\alpha$ *has approximation ratio larger than $\beta_d d$.*

Moreover,
$\beta_3 = 2/3 \approx 0.66667$,
$\beta_4 = (\sqrt{13} - 1)/4 \approx 0.65139$,
$\beta_d = 2^{-1/\log_2 3} \approx 0.64576$ *for* $d \geq 5$.
Algorithm 2-IMP$^\alpha$ *can be modified to a* $(\frac{k}{k-1})\beta_d d$-*approximation with running time of* $O(k^\alpha (dn)^{2+\alpha})$ *for any integer* $k > 1$.

3 Weight transfer and help

To prove an upper bound $d\beta$ ($\beta = \beta_d$) on the approximation ratio, it suffices to do the following: given set A that is obtained when the algorithm 2-IMP$^\alpha$ terminates, and set A^* that maximizes $w(A^*)$, we transfer the weights of nodes in A^* to their neighbors in A in such a way that no node $v \in A$ receives more than $d\beta w(v)$, or, in other words, that the average transfer to v from its neighbors is at most $\beta w(v)$.

We conduct the transfer in two rounds. Let γ be defined by the equality

$$(d-1)(\beta - \gamma) = 1 - \beta.$$

In round one, $u \in A^*$ sends to each $v \in N(u, A)$ exactly $\gamma w(v)$. As a result, u is left with

$$excess(u) = w(u) - \gamma w(N(u, A))$$

to be distributed in round two. (If $excess(u) \leq 0$, u does not participate in round two.) On the receiving end, each $v \in A$ gets from each of its neighbors in A^* exactly $\gamma w(v)$, so it suffices if in round two it receives from elements of $N(v, A^*)$, on average, $(\beta - \gamma)w(v)$ or less.

The following quantity will allow us to control the size of the transfers: if $u \in A^*$ and $v \in N(u, A)$, then

$$help(u, v) = w^\alpha(u) - w^\alpha(N(u, A - v))$$

and the help coefficient is

$$h(u, v) = \frac{help(u, v)}{w^\alpha(v)}.$$

Similarly, if $trans(u, v)$ is the transfer of weight from u to v in round two, then the transfer coefficient is

$$t(u, v) = \frac{trans(u, v)}{w(v)}.$$

LEMMA 3.1. *Assume that* 2-IMP$^\alpha$ *has terminated,* $v \in A$*, and that* u_0 *and* u_1 *are two different elements of* $N(v, A^*)$*. Then* $h(u_0, v) + h(u_1, v) \leq 1$.

Proof. Suppose that $h(u_0, v) + h(u_1, v) > 1$. Then $\frac{w^\alpha(u_0) - w^\alpha(N(u_0, A-v))}{w^\alpha(v)} + \frac{w^\alpha(u_1) - w^\alpha(N(u_1, A-v))}{w^\alpha(v)} > 1 \equiv$ $w^\alpha(u_0) - w^\alpha(N(u_0, A - v)) + w^\alpha(u_1) - w^\alpha(N(u_1, A - v)) > w^\alpha(v) \equiv w^\alpha(\{u_0, u_1\}) > w^\alpha(N(u_0, A - v)) + w^\alpha(N(u_1, A - v)) + w^\alpha(v) \Rightarrow w^\alpha(\{u_0, u_1\}) > w^\alpha(N(\{u_0, u_1\}, A)$. The latter means that $\{u_0, u_1\}$ improves $w^\alpha(A)$, a contradiction because we assumed that the algorithm 2-IMP$^\alpha$ has terminated. ❑

We will say that $u \in A^*$ *provides adequate help to* $v \in N(u, A)$ if the following holds for $t = t(u, v)$ and $h = h(u, v)$:

$$h \geq \begin{cases} 1 - (1 - (d-1)t)^\alpha & \text{if } t \leq \beta - \gamma, \\ (\gamma + t)^\alpha & \text{if } t \geq \beta - \gamma. \end{cases}$$

Observe that for $t = \beta - \gamma$ the right-hand-side is continuous: $1 - (1 - (d-1)(\beta - \gamma))^\alpha = 1 - (1 - (1 - \beta))^\alpha = 1 - \beta^\alpha = 1 - 1/2 = 1/2$, and $(\gamma + \beta - \gamma)^\alpha = \beta^\alpha = 1/2$. It is easy to show that adequate help indeed assures our result.

LEMMA 3.2. *Assume that* 2-IMP$^\alpha$ *has terminated,* $v \in A$ *and every* $u \in N(v, A^*)$ *provides* v *with adequate help. Then*

$$\sum_{u \in N(v, A^*)} trans(u, v) \leq (\beta - \gamma) d w(v).$$

Proof. To simplify the calculation, we divide all node weights by $w(v)$; then $w(v) = 1$, $help(u, v) = h(u, v)$ and $trans(u, v) = t(u, v)$, and our claim is

$$\sum_{u \in N(v, A^*)} t(u, v) \leq (\beta - \gamma) d.$$

We may assume that $N(v, A^*) = \{u_1, \cdots, u_d\}$, $t_i = t(u_i, v)$ and that t_1 has the maximal value among t_i's, we will also use h_i to denote $h(u_i, v)$. If $t_1 \leq \beta - \gamma$, we are done. Otherwise, for some $x > 0$ we have $t_1 = \beta - \gamma + (d-1)x$. It suffices to show that $t_i \leq \beta - \gamma - x$ for $1 < i \leq d$.

Suppose that the contrary holds for t_2. Then
$h_1 + h_2 >$
$(\beta + (d-1)x)^\alpha + 1 - (1 - (d-1)(\beta - \gamma - x))^\alpha =$
$(\beta + (d-1)x)^\alpha + 1 - (1 - (1 - \beta) - (d-1) - x))^\alpha =$
$(\beta + (d-1)x)^\alpha + 1 - (\beta + (d-1)x)^\alpha = 1$. Thus
$h_1 + h_2 > 1$, a contradiction with Lemma 3.1. ❑

For all $d \geq 5$ we will be using the value of β computed in Lemma 2.4. We will also use the same value of γ and the same definition of adequate help, namely those computed for $d = 5$. Then Lemma 3.2 still applies to $d > 5$. If we closely inspect the proof of that lemma, it shows that for each $v \in A$, the average of the top five transfer coefficients to node v is at most $\beta - \gamma$. Because at most one transfer coefficient can be larger than $\beta - \gamma$, the average for the remaining transfer coefficients is at most $\beta - \gamma$ as well. Therefore it will suffice that for every d the transfers from A^* to A are provided with adequate help.

4 Weight transfer with more than two neighbors

In round 2, a node $u \in A^*$ transfers its excess weight either to one selected neighbor in A that has the maximum weight, or to two neighbors with maximal weights. In this section we will show that it suffices to analyze the cases when $|N(u, A)| \leq 2$.

LEMMA 4.1. *If* $x, y, z > 0$ *and* $a > 1$*, then* $(x + y + z)^a - (x + z)^a > (x + y)^a - x^a$.

Proof. We need to show that $(x+y+z)^a - (x+z)^a$ is an increasing function of z. The derivative over z, after dividing by factor a, equals $(x + y + z)^{a-1} - (x + z)^{a-1}$. This follows from the fact that $(x + y + z)^{a-1}$ is an increasing function of z. ❑

Suppose first that $d = 3$; note that then $\gamma = 1/2$.

LEMMA 4.2. *Assume that* $u \in A^*$*,* $\{v_0, v_1, v_2\} \subseteq N(u, A)$*,* $w_* = w(u)$*,* $w_i = w(v_i)$ *and* $w_0 \geq w_1 \geq$

w_2. *Then after we decrease $w(v_2)$ by w_2 and $w(u)$ by $\gamma w_2 = w_2/2$, excess(u) remains unchanged, and neither help(u, v_0) nor help(u, v_1) increases.*

Proof. That *excess*(u) does not change, it follows from the definition. In the formula for *help*(u, v_i) we replace $w_*^\alpha - w_2^\alpha$ with $(w_* - w_2/2)^\alpha$. Thus we need to show that

$$w_*^\alpha - w_2^\alpha \geq (w_* - w_2/2)^\alpha \equiv w_*^\alpha - (w_* - w_2/2)^\alpha \geq w_2^\alpha.$$

By Lemma 4.1, $w_*^\alpha - (w_* - w_2/2)^\alpha$ is smallest when w^* is smallest possible. Because *excess*$(u) > 0$ we have $w_* > {}^1\!/_2\, w(N(u, A)) \geq {}^3\!/_2\, w_2$. Thus

$$w_*^\alpha - (w_* - w_2/2)^\alpha \geq ({}^3\!/_2\, w_2)^\alpha - w_2^\alpha = 2w_2^\alpha - w_2^\alpha = w_2^\alpha.$$

❑

When $d \geq 3$ is arbitrary, the argument is similar to this in Lemma 4.2, and is omitted here.

LEMMA 4.3. *Assume that $u \in A^*$, $\{v_0, v_1, v_2, \ldots, v_{d-1}\} \subseteq N(u, A)$, $w_* = w(u)$, $w_i = w(v_i)$ and $w_0 \geq w_1 \geq w_2 \geq \ldots \geq w_{d-1}$. Then after we decrease $w(v_{d-1})$ by w_{d-1} and $w(u)$ by γw_{d-1}, excess(u) remains unchanged, and none of help(u, v_i), for $i = 0, 1, \ldots, d-2$, increases.*

Below we will keep the notation $u, v_0, v_1, w_*, w_0, w_1$ that was introduced here.

5 Weight transfer from a small node

In this section we assume that $w_* \leq w_0$. In this case *help*$(u, v_1) = w_*^\alpha - w_0^\alpha \leq 0$ and thus u transfers its entire excess weight to v_0. We may assume that $w_* > w_1$, otherwise *excess*$(u) = w_* - {}^1\!/_2\,(w_0 + w_1) \leq 0$.

LEMMA 5.1. *Assume that $u \in A^*$, $v \in N(u, A)$, v has the largest weight in $N(u, A)$ and $w(u) \leq w(v)$. Then u provides adequate help to v.*

Proof. Like in the proof of Lemma 3.2, we rescale w in such a way that $w_0 = 1$, and thus $w_* \leq 1$. We define x such that $w_1 = 2x$ and t such that $w_* = {}^1\!/_2 + x + t$. One can see that t is the transfer of weight from u to v, as well as the transfer coefficient. The help (and the help coefficient) that u provides to v equals $h_t(x) = ({}^1\!/_2 + t + x)^\alpha - (2x)^\alpha$. According to our assumptions, ${}^1\!/_2 + x + t \leq 1$, equivalently, $0 \leq x \leq {}^1\!/_2 - t$.

A simple calculus (omitted in the extended abstract) shows that the second derivative of $h_t(x)$ is negative for $x < {}^1\!/_2$. Therefore we can provide the following lower estimate for the help coefficient that is associated with transfer coefficient t:

$$h(t) = \min_{0 \leq x \leq {}^1\!/_2 - t} h_t(x) = \min(h_t(0), h_t({}^1\!/_2 - t))$$

$$= \min(({}^1\!/_2 + t)^\alpha, 1 - (1 - 2t)^\alpha).$$

Now we can establish which of the two quantities in the latter min is smaller. Note that as a function of t, both are defined in the entire interval $[0, {}^1\!/_2]$, the second derivative of the first quantity is strictly positive, while that of the second quantity is strictly negative. Thus if these quantities are equal for some two values t_0, t_1, then the first quantity is smaller in the interval (t_0, t_1).

Note that $t_0 = {}^1\!/_6$, because $({}^1\!/_2 + {}^1\!/_6)^\alpha = ({}^2\!/_3)^\alpha = {}^1\!/_2 = 1 - (1 - 2 \times {}^1\!/_6)^\alpha$. Similarly, $t_1 = {}^1\!/_2$, because $({}^1\!/_2 + {}^1\!/_2)^\alpha = 1 - (1 - 2 \times {}^1\!/_2)^\alpha$.

We can conclude that $h(t)$ is $1 - (1 - 2t)^\alpha$ for $t \leq {}^1\!/_6$ and $({}^1\!/_2 + t)^\alpha$ for $t \geq {}^1\!/_6$. ❑

6 Weight transfer from a large node, $d = 3$

In this section we assume that $w_* \geq w_0$. Again, assume that $w_0 = 1$, and assume also that $w_1 = 1 - 2x$, $w_* = 1 + t - x$. We define the weight transfer as follows:

$$\begin{aligned} \textit{trans}(u, v_0) &= {}^1\!/_2\,(t + x), \\ \textit{trans}(u, v_1) &= {}^1\!/_2\,(t - x). \end{aligned}$$

First we will show that v_0 receives adequate help.

Case: ${}^1\!/_2\,(t + x) \leq {}^1\!/_6$. We need to show that

$$(1 + t - x)^\alpha - (1 - 2x)^\alpha \geq 1 - (1 - (t + x))^\alpha.$$

Since $1 + t - x \geq 1$, $(1 + t - x) - (1 - 2x) = 1 - (1 - (t + x))$, this inequality follows by Lemma 4.1.

Case: ${}^1\!/_2\,(t + x) \geq {}^1\!/_6$. We need to show that

$$(1 + t - x)^\alpha - (1 - 2x)^\alpha \geq ({}^1\!/_2 + {}^1\!/_2\,(t + x))^\alpha.$$

A change $(t, x) \leftarrow (t - \delta, x + \delta)$ does not change the right-hand-side but it decreases the left-hand-side by Lemma 4.1. Thus it suffices to consider the case when this change cannot be applied, *i.e.* $t = x$:

$$1 - (1 - 2x)^\alpha \geq ({}^1\!/_2 + x)^\alpha \;\equiv\; (1 - 2x)^\alpha + ({}^1\!/_2 + x)^\alpha \leq 1.$$

According to our assumptions, ${}^1\!/_2\,(t + x) = x \geq {}^1\!/_6$ and $1 - 2x \geq 0$, thus ${}^1\!/_6 \leq x \leq {}^1\!/_2$. Because the last left-hand-side is a convex function of x, it suffices to check this inequality for $x = {}^1\!/_6$ (${}^1\!/_2 + {}^1\!/_2 \leq 1$) and for $x = {}^1\!/_2$ $(0 + 1 \leq 1)$. Similarly, we can show that v_1 also receives adequate help. In this case the coefficients of transfer and help are

$$\tau = \frac{1}{2}\frac{t - x}{1 - 2x}, \qquad h = \frac{(1 + t - x)^\alpha - 1}{(1 - 2x)^\alpha}.$$

Case: $\tau \leq {}^1\!/_6$. We need to show that $h \geq 1 - (1 - 2\tau)^\alpha$, *i.e.*

$$\frac{(1 + t - x)^\alpha - 1}{(1 - 2x)^\alpha} \geq \frac{(1 - 2x)^\alpha - (1 - 2x - (t - x))^\alpha}{(1 - 2x)^\alpha}$$

and this follows immediately from Lemma 4.1.
Case: $\tau \geq 1/6$. We need to show that $h \geq (1/2 + \tau)^\alpha$, *i.e.*

$$\frac{(1+t-x)^\alpha - 1}{(1-2x)^\alpha} \geq \left(\frac{1}{2} + \frac{1}{2}\frac{t-x}{1-2x}\right)^\alpha =$$

$$\frac{1}{2^\alpha}\frac{(1+t-3x)^\alpha}{(1-2x)^\alpha} \equiv 2^\alpha(1+t-x)^\alpha - (1+t-3x)^\alpha \geq 2^\alpha.$$

The left hand side is an increasing function of t, so it suffices to consider the smallest possible t:

$$\frac{1}{2}\frac{t-x}{1-2x} \geq \frac{1}{6} \quad \equiv \quad 3t - 3x \geq 1 - 2x \quad \equiv \quad t \geq \frac{1+x}{3}.$$

Thus it suffices to show $2^\alpha\left(\frac{4-2x}{3}\right)^\alpha - \left(\frac{4-8x}{3}\right)^\alpha \geq 2^\alpha \equiv (2-x)^\alpha - (1-2x)^\alpha \geq \left(\frac{3}{2}\right)^\alpha = 2$. The last left-hand-side is a concave function of x, thus it suffices to check the inequality for the largest and the smallest possible values of x, *i.e.* for $x = 0,\ 1/2$. This concludes the analytical proof of the ratio of $2/3\,d$ for our algorithm for all values of d. This also proves the part of Theorem 2.1 in case of $d = 3$.

The next section establishes the optimum approximations ratios of 2-IMP$^\alpha$ in cases when $d \geq 4$.

7 Weight transfer from a large node, $d > 3$

In this section we assume that $w_* \geq w_0$, $w_0 = 1$, and assume also that $w_1 = x$, $w_* = y$. In the first round node $u \in A^*$ transfers γw_i to node $v_i, i \in \{0,1\}$, so the total transfer is $\gamma(x+1)$. The excess to be transfered in the second round is $y - \gamma(x+1)$. Unlike in section 6, we will not explicitly specify amounts of the excess that will go to node v_0 and to v_1. We prove that there is enough total help (i.e. the help provided by node u to v_0 and to v_1) so that the total excess can be transfered to these nodes instead. This will of course imply that there is a split of the excess to nodes v_0, v_1 that will work as in section 6.

For $u \in A^*$, $v \in N(u,A)$, and $t = t(u,v)$, $h = h(u,v)$ the formula for the adequate help is:

$$h \geq \begin{cases} 1 - (1-(d-1)t)^\alpha & \text{if } t \leq \beta - \gamma \\ (\gamma + t)^\alpha & \text{if } t \geq \beta - \gamma \end{cases}$$

Since we have assumed earlier that h as a function of t is continuous, we have that $1 - (1-(d-1)(\beta - \gamma))^\alpha = (\gamma + \beta - \gamma)^\alpha$, and by $\beta^\alpha = 1/2$, we have $(1-(d-1)(\beta-\gamma))^\alpha = 1/2$, and $1-(d-1)(\beta-\gamma) = \beta$. Finally,

$$\gamma = \frac{d\beta - 1}{d-1}. \tag{7.1}$$

Using the formula for h, we can compute transfer as a function of help as follows.

$$t = t(h) = \begin{cases} \frac{1}{(d-1)}(1-(1-h)^{1/\alpha}) & \text{if } h \leq 1/2, \\ h^{1/\alpha} - \gamma & \text{if } h \geq 1/2. \end{cases} \tag{7.2}$$

Then, by our parametrization, the help coefficient to the small weight node v_1 is $h_1 = h(u, v_1) = \frac{y^\alpha - 1}{x^\alpha}$, and same for the large weight node v_0 is $h_0 = h(u, v_0) = y^\alpha - x^\alpha$. The total allowed transfer to node v_1 given help h_1 is $allowed_s(x,y) = x \cdot t\left(\frac{y^\alpha - 1}{x^\alpha}\right)$ (we have to multiply the transfer coefficient by $w(v_1) = x$ to obtain the transfer). Also, the total allowed transfer to node v_0 given help h_0 is $allowed_l(x,y) = t\,(y^\alpha - x^\alpha)$.

To prove the claim, we need to show that for appropriate ranges of x and y we have $required(x,y) = y - \gamma(x+1) \leq allowed_l(x,y) + allowed_s(x,y)$, that is

$$required(x,y) \;\leq\; allowed_l(x,y) + allowed_s(x,y), \tag{7.3}$$

which is

$$y - \gamma(x+1) \;\leq\; t\,(y^\alpha - x^\alpha) + x \cdot t\left(\frac{y^\alpha - 1}{x^\alpha}\right). \tag{7.4}$$

Letting $\phi(x,y) = allowed_l(x,y) + allowed_s(x,y) - required(x,y)$, we want to prove that $\phi(x,y) \geq 0$ for appropriate x, y.

By our assumption, v_0 was the largest weight neighbor of u and $w(v_0) = 1$. The other neighbor v_1 of u has therefore weight $w(v_1) = x \leq 1$. And, the weight of u was larger than $w(v_0)$, thus $w(u) = y \geq 1$. Also, help to large node $y^\alpha - x^\alpha$ is at most one, thus $y \leq 2$. So the ranges are $(x,y) \in [0,1] \times [1,2]$.

We finish the proof that $\phi(x,y) \geq 0$ by verifying it on the computer for most of the cases and by computing partial derivatives in the remaining special cases. Details are in the Appendix. Given that we perform separate analysis for $d = 4$ and $d > 4$, this concludes the proof of Theorem 2.1.

8 Conclusions

We have presented a family of local search approximation algorithms for w-MIS on $(d+1)$-claw free graphs. Our algorithms improve the approximation ratio attainable in polynomial time that is independent from d—from $d-1$ to $2/3\,d$ and less. We have found the optimum value of the parameters for our algorithms and we have established exactly the approximation ratios.

Many open problems remain. Consider a family of algorithms like 2-IMP$^\alpha$ but with an arbitrary function $f(w)$ instead of function w^α. What would be the best choice of f ? Can one get a better ratio by using several such functions and then taking the best result ? How do other local search methods like 3-IMP$^\alpha$?

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A Computer-assisted proof

We have written a very simple C program to prove inequality (7.4) within the ranges of x, y. There are cases where the program might not succeed in proving (7.4). In those cases we give analytical proofs. The ideas behind the program are as follows.

Each term in (7.4): $required(x, y) = y - \gamma(x+1)$, $allowed_l(x, y) = t(y^\alpha - x^\alpha)$, and $allowed_s(x, y) = x \cdot t\left(\frac{y^\alpha - 1}{x^\alpha}\right)$ as a function of x, y is increasing in y and decreasing in x. This is obvious for the first two terms, and can easily be proven for the third term by using for instance partial derivatives. The program starts with the whole square $[0,1] \times [1,2]$ to test, the coordinates (with vertical x-axis, horizontal y-axis) of the corners being:

- left lower corner (x, y), right lower corner $(x, y+a)$,
- left upper corner $(x+a, y)$, right upper corner $(x+a, y+a)$.

Here a is the current width of the square. At the beginning: $x = 0$, $y = 1$, and $a = 1$, which corresponds to the square $[0,1] \times [1,2]$.

Having the current values of x, y, a, the largest possible value of $required(\cdot,\cdot)$ in the current square is $required(x, y+a)$, and the smallest possible values of $allowed_l(\cdot,\cdot)$ and of $allowed_s(\cdot,\cdot)$ are $allowed_l(x+a, y)$ and $allowed_s(x+a, y)$. This follows from the monotonicity properties discussed above. Thus if we have that

$$required(x, y+a) \le allowed_l(x+a, y) + allowed_s(x+a, y),$$

then (7.3) is true in the whole current square. Otherwise, the program does not succeed in the current square. In this case we set $a := a/2$ and recurse on the four subsquares of the current square. We do this until reaching some prespecified accuracy of a. Our program outputs the success squares, and the unsuccessful ones into a text file. There are small number of very small unsuccessful areas. Within unsuccessful areas, we set up some distance around these areas and prove (7.3) analytically there. These proofs will be presented subsequently.

We first observe that

$$\frac{y^\alpha - 1}{x^\alpha} \le y^\alpha - x^\alpha \quad \Leftrightarrow \quad y^\alpha - 1 \le x^\alpha y^\alpha - x^{2\alpha} \quad \Leftrightarrow$$
$$y^\alpha(1 - x^\alpha) \le (1 + x^\alpha)(1 - x^\alpha) \quad \Leftrightarrow \quad y^\alpha \le 1 + x^\alpha,$$

where we have used that $1 - x^\alpha \ge 0$. The last inequality holds as discussed above.

We divide now the range $[0,1] \times [1,2]$ into subareas according to the following.

(a) $y > \gamma(x+1)$ (i.e. positive transfer in 2nd round) and $y^\alpha - x^\alpha \le 1/2$,

(b) $y^\alpha - x^\alpha > 1/2$ and $\frac{y^\alpha - 1}{x^\alpha} \le 1/2$,

(c) $\frac{y^\alpha - 1}{x^\alpha} > 1/2$ and $y^\alpha - x^\alpha \le 1$.

Using the observation above, and formula (7.2) we obtain the following expressions for function ϕ depending on cases (a)-(c):

- Case (a): $\phi(x,y) = \frac{1}{d-1}(1-(1-(y^\alpha - x^\alpha))^{1/\alpha}) + \frac{x}{d-1}\left(1-\left(1-\frac{y^\alpha-1}{x^\alpha}\right)^{1/\alpha}\right) - y + \gamma(x+1)$,
- Case (b): $\phi(x,y) = (y^\alpha - x^\alpha)^{1/\alpha} - \gamma + \frac{x}{d-1}\left(1-\left(1-\frac{y^\alpha-1}{x^\alpha}\right)^{1/\alpha}\right) - y + \gamma(x+1)$,
- Case (c): $\phi(x,y) = (y^\alpha - x^\alpha)^{1/\alpha} - \gamma + x\left(\frac{y^\alpha-1}{x^\alpha}\right)^{1/\alpha} - \gamma x - y + \gamma(x+1)$,

which can be simplified to the following

- (a): $\phi(x,y) = \frac{1}{d-1}(1-(1-y^\alpha+x^\alpha)^{1/\alpha}) + \frac{1}{d-1}(x-(1-y^\alpha+x^\alpha)^{1/\alpha}) - y + \gamma(x+1)$,
- (b): $\phi(x,y) = (y^\alpha - x^\alpha)^{1/\alpha} + \frac{1}{d-1}(x-(1-y^\alpha+x^\alpha)^{1/\alpha}) - y + \gamma x$,
- (c): $\phi(x,y) = (y^\alpha - x^\alpha)^{1/\alpha} + (y^\alpha - 1)^{1/\alpha} - y$.

It turns out that to obtain a tight characterization of the approximation ratio we have to separately deal with the case of $d = 4$ and with the case when $d \geq 5$. However, by the discussion in the last paragraph of Section 3, in case of $d \geq 5$ it suffices to prove the claim only for $d = 5$. We first collect the formulas for parameters α, β and γ. They follow from the considerations in the previous sections.

Case of $d = 4$. In this case the tight ratio in characterized through Lemmas 2.1 and 2.2. If we make the two expressions there equal, then this gives $\beta = \frac{\sqrt{5d^2-8d+4}+2-d}{2d} \approx 0.651388$. Then since $\beta = (1/2)^{1/\alpha}$, $\alpha = \log(0.5)/\log(\beta) \approx 1.617047$, and by (7.1) $\gamma = \frac{d\beta-1}{d-1} \approx 0.535184$.

Case of $d = 5$. Here we use Lemma 2.4. We have

$$\beta = (1/2)^{1/\alpha} = (3/2)^{1/\alpha}/2 \quad \text{and} \quad \alpha = log_2 3,$$

which gives $\alpha \approx 1.584963$, $\beta \approx 0.64576$, and $\gamma = \frac{d\beta-1}{d-1} \approx 0.5572$.

In the remainder of this section we present the analytical proofs of $\phi(x,y) \geq 0$ in the unsuccessful areas in cases (a)-(c), treating simultaneously the cases of $d = 4$ and $d = 5$.

A.1 Unsuccessful areas in case (a) In this case our program does not succeed in a close area around point $(x,y) = (\beta, 1)$ (in case of $d = 4$) and $(x,y) = (1, 2\beta)$ (in case of $d = 5$). A safe margin around these points as obtained by the program can be set to $\delta = 0.003$. This means that we have to prove that $\phi(x,y) \geq 0$ for:

- $\beta \leq x \leq \beta + \delta$ and $1 \leq y \leq 1 + \delta$ (when $d = 4$). It is enough to consider $x \geq \beta$ since by (a) $y^\alpha - x^\alpha \leq 1/2$, and so $1/2 = 1 - 1/2 \leq y^\alpha - 1/2 \leq x^\alpha$ which gives $\beta = (1/2)^{1/\alpha} \leq x$.
- $1 - \delta \leq x \leq 1$ and $2\beta - \delta \leq y \leq 2\beta$ (when $d = 5$). It is enough to consider $y \leq 2\beta$ since by (a) $y^\alpha - x^\alpha \leq 1/2$, and so $2\beta = (3/2)^{1/\alpha} \geq (1/2 + x^\alpha)^{1/\alpha} \geq y$.

Please note, that we have to use different formulas/values for α, β, γ considering case $d = 4$ and case $d = 5$. We treat these two cases together since the formula for function ϕ is the same. The reason that our program fails around those two points is that one can using exact formulas for α, β, γ easily show that $\phi(\beta, 1) = 0$ $(d = 4)$ and $\phi(1, 2\beta) = 0$ $(d = 5)$.

One can show that the partial derivative of ϕ on y is as follows.

$$\frac{\partial \phi}{\partial y} = \frac{2}{d-1} y^{\alpha-1}(1-y^\alpha+x^\alpha)^{(1/\alpha)-1} - 1.$$

Since $\alpha > 1$ $(d = 4, 5)$, both functions $y^{\alpha-1}$ and $(1-y^\alpha+x^\alpha)^{(1/\alpha)-1}$ are increasing as y increases, and so does the derivative $\frac{\partial \phi}{\partial y}$. In case (a), $y^\alpha - x^\alpha \leq 1/2$, so $(1/2 + x^\alpha)^{1/\alpha} \geq y$, and we can bound the derivative as.

$$\frac{\partial \phi}{\partial y} \leq \frac{2}{d-1} y^{\alpha-1}(1/2)^{(1/\alpha)-1} - 1 = \frac{4}{d-1} y^{\alpha-1}\beta - 1.$$

We claim that in both cases $d = 4, 5$, $\frac{\partial \phi}{\partial y} < 0$.
$d = 4$: We find the largest y such that $\frac{\partial \phi}{\partial y} \leq \frac{4}{d-1} y^{\alpha-1}\beta - 1 \leq 0$. We have $y \leq (3/(4\beta))^{1/(\alpha-1)} \approx 1.25666$, which of course is true since in our case $y \leq 1 + \delta = 1.003$. (By the conditions of case (a) we can show that $y \leq 2\beta \approx 1.3033$, thus this estimate on $\frac{\partial \phi}{\partial y}$ is not sufficient for a general proof in case (a).)

$d = 5$: By $y^\alpha \leq x^\alpha + 1/2$, $x \leq 1$, and by the estimate above we obtain: $\frac{\partial \phi}{\partial y} \leq \frac{4}{d-1} y^{\alpha-1}\beta - 1 \leq \frac{4}{d-1}(x^\alpha + 1/2)^{1-1/\alpha}\beta - 1 \leq \frac{4}{d-1}(3/2)^{1-1/\alpha}\beta - 1$. Thus, $\frac{\partial \phi}{\partial y} \leq ((3/2)^{1/\alpha})^{\alpha-1}\beta - 1 = (2\beta)^{\alpha-1}\beta - 1 = (2\beta)^\alpha \frac{1}{2} - 1 = \frac{3}{2}\frac{1}{2} - 1 = -1/4 < 0$.

Therefore, for any fixed x, function $\phi(x,\cdot)$ is decreasing, and so it takes the smallest value for the largest possible y, which is $(x^\alpha + 1/2)^{1/\alpha}$. From now on we can focus on function $\psi(x) = \phi(x, (x^\alpha + 1/2)^{1/\alpha})$ and show that it is non-negative. We obtain that $\psi(x) = \frac{1-\beta}{d-1} + \frac{x-\beta}{d-1} - (1/2 + x^\alpha)^{1/\alpha} + \gamma(x+1)$, and $\psi'(x) = \frac{1}{d-1} + \gamma - (\frac{1}{2x^\alpha} + 1)^{(1/\alpha)-1}$. Since $\alpha > 1$ (in both cases $d = 4, 5$), $(1/\alpha) - 1 < 0$, and we can easily see that ψ' decreases as x increases, which means that function ψ is concave. Thus to show that $\psi(x) \geq 0$ it suffices to check this only in the endpoints of an appropriate interval.

Check for $d = 4$. Here $\beta \leq x \leq \beta + \delta$. We have $\psi(\beta) = \phi(\beta, 1) = 0$ (this can be shown by using exact formulas for α, β, γ). Also, $\psi(\beta + \delta) = \frac{1-\beta}{d-1} + \frac{\delta}{d-1} - (1/2 + (\beta+\delta)^\alpha)^{1/\alpha} + \gamma(\beta + \delta + 1) \approx 0.00030148 > 0$.
Check for $d = 5$. Here $1 - \delta \leq x \leq 1$. We have $\psi(1) = \phi(1, 2\beta) = 0$ (this also can be shown by using exact formulas for α, β, γ). Now, $\psi(1-\delta) = \frac{1-\beta}{d-1} + \frac{1-\delta-\beta}{d-1} - (1/2 + (1-\delta)^\alpha)^{1/\alpha} + \gamma(2-\delta) \approx 0.00016032 > 0$.

A.2 Unsuccessful areas in case (b) In this case our program does not succeed in a close area around points $(0,1), (\beta,1)$ ($d=4$) and $(0,1), (1,2\beta)$ (for $d=5$). A safe margin around these points as obtained by the program can be set to $\delta = 0.003$. This means that we have to prove that $\phi(x,y) \geq 0$ for:

- $0 \leq x \leq \delta$ and $1 \leq y \leq 1+\delta$ (in both cases $d = 4, 5$).
- $\beta - \delta \leq x \leq \beta + \delta$ and $1 \leq y \leq 1 + \delta$ (when $d = 4$).
- $1 - \delta \leq x \leq 1$ and $2\beta - \delta \leq y \leq 2\beta$ (when $d = 5$). It is enough to consider $y \leq 2\beta$ since by (b) $(y^\alpha - 1)/x^\alpha \leq 1/2$, and so $2\beta = (3/2)^{1/\alpha} \geq (1 + x^\alpha/2)^{1/\alpha} \geq y$.

The reason that our program fails around those two points is that one can again using exact formulas for α, β, γ easily show that $\phi(0,1) = 0$ ($d = 4,5$), $\phi(\beta,1) = 0$ ($d=4$), and $\phi(1,2\beta) = 0$ ($d=5$).

The partial derivative of ϕ is as follows: $\frac{\partial \phi}{\partial y} =$

$= y^{\alpha-1}(y^\alpha - x^\alpha)^{(1/\alpha)-1} + \frac{y^{\alpha-1}}{d-1}(1 - y^\alpha + x^\alpha)^{(1/\alpha)-1} - 1.$

Since $\alpha > 1$ ($d = 4,5$), we have that $(1/\alpha) - 1 < 0$, and therefore $y^{\alpha-1}(y^\alpha - x^\alpha)^{(1/\alpha)-1} = (1 - x^\alpha/y^\alpha)^{(1/\alpha)-1}$ is decreasing as y increases. $(1 - y^\alpha + x^\alpha)^{(1/\alpha)-1}$ is decreasing as a function of y. Finally, the first term in $\frac{\partial \phi}{\partial y}$ is decreasing, and the second term is increasing as a function of y. We use this to lower bound $\frac{\partial \phi}{\partial y}$. Since in case (b) we have $(1/2 + x^\alpha)^{1/\alpha} < y \leq (1 + x^\alpha/2)^{1/\alpha}$, we can lower bound $\frac{\partial \phi}{\partial y}$ as follows: $\frac{\partial \phi}{\partial y} > (1 + x^\alpha/2)^{(\alpha-1)/\alpha}(1 - x^\alpha/2)^{(1/\alpha)-1} + \frac{1}{d-1}(1/2 + x^\alpha)^{(\alpha-1)/\alpha}(1/2)^{(1/\alpha)-1} - 1$. Observe now that all the expressions above are increasing functions of x (recall that $(1/\alpha) - 1 < 0$). The smallest value of x is zero for both cases $d = 4,5$. This gives;

$$\frac{\partial \phi}{\partial y} > 1 + \frac{(1/2)^{(\alpha-1)/\alpha}(1/2)^{(1/\alpha)-1}}{d-1} - 1 = \frac{1}{d-1} > 0.$$

Thus we have the following.

LEMMA A.1. *In case (b), we have $\frac{\partial \phi}{\partial y} > 0$ for any fixed x, and any y within the conditions of (b).*

We now focus on the point $(0,1)$ in both cases $d = 4,5$. In this case $0 \leq x \leq \delta$, and by (b) we have the following lower bounds on y: $y > (1/2 + x^\alpha)^{1/\alpha}$ and $y \leq 1$. Thus the lower bound $y \leq 1$ prevails, and so 1 is the smallest possible value for y. By Lemma A.1, for any fixed x, function ϕ is increasing with y. So the smallest value is for $y = 1$. Thus it suffices to focus on a function $\psi_1(x) = \phi(x,1)$. We have that $\psi_1(x) = (1 - x^\alpha)^{1/\alpha} + \gamma x - 1$. Notice, that $\psi_1(0) = \phi(0,1) = 0$. Thus to prove that $\psi_1(x) \geq 0$ when $0 \leq x \leq \delta$, it suffices to show that $\psi_1'(x) \geq 0$.

The derivative of ψ_1 is $\psi_1'(x) = \gamma - x^{\alpha-1}(1 - x^\alpha)^{(1/\alpha)-1}$. Since $(1/\alpha) - 1 < 0$, $(1 - x^\alpha)^{(1/\alpha)-1}$ and also $x^{\alpha-1}$ increase as x increases. Thus, $\psi_1'(x)$ is a decreasing function. Now, if $d = 4$ then we can show that $\psi_1'(0.3) \approx 0.03066 > 0$, and so we have for any $x \in [0,\delta]$, $\psi_1'(x) \geq \psi_1'(\delta) = \psi_1'(0.003) \geq \psi_1'(0.3) > 0$. Similarly, if $d = 5$, then we can show that $\psi_1'(0.3) \approx 0.03254 > 0$, and so again for any $x \in [0,\delta]$, $\psi_1'(x) > 0$.

Now we turn our attention to the point $(\beta, 1)$ ($d = 4$). By the conditions of (b) we have that $y > (1/2 + x^\alpha)^{1/\alpha}$, and $y \geq 1$.

Suppose first that $\beta \leq x \leq \beta + \delta$. Then, $(1/2 + x^\alpha)^{1/\alpha} \geq (1/2 + \beta^\alpha)^{1/\alpha} = 1$, so the case $y > (1/2 + x^\alpha)^{1/\alpha}$ prevails (over $y \geq 1$) and so by Lemma A.1, it suffices to show that $\psi_2(x) = \phi(x, (1/2 + x^\alpha)^{1/\alpha}) \geq 0$, for appropriate x's. Thus, $\psi_2(x) = (1/2)^{1/\alpha} + \frac{1}{d-1}(x - 1/2)^{1/\alpha}) - (1/2 + x^\alpha)^{1/\alpha} + \gamma x = \beta + \gamma x + \frac{1}{d-1}(x - \beta) - (1/2 + x^\alpha)^{1/\alpha}$. The derivative is $\psi_2'(x) = \gamma + \frac{1}{d-1} - x^{\alpha-1}(1/2 + x^\alpha)^{(1/\alpha)-1} = \gamma + \frac{1}{d-1} - (\frac{1}{2x^\alpha} + 1)^{(1/\alpha)-1}$. It is easy to see that $\psi_2'(x)$ is a decreasing function of x. Now, $\psi_2'(1) = \gamma + \frac{1}{d-1} - (3/2)^{(1/\alpha)-1} \approx 0.01186 > 0$, therefore $\psi_2'(x) > 0$ for all $x \in [\beta, \beta + \delta]$. Since $\psi_2(\beta) = \phi(\beta, 1) = 0$ and ψ_2 is increasing, the claim follows.

Suppose now that $\beta - \delta \leq x \leq \beta$. Then the case $y \geq 1$ prevails over $y > (1/2 + x^\alpha)^{1/\alpha}$ and by Lemma A.1, it suffices to show that $\psi_1(x) = \phi(x,1) \geq 0$. By the argument above $\psi_1'(x) = \gamma - x^{\alpha-1}(1-x^\alpha)^{(1/\alpha)-1}$ is decreasing. Now, $\psi_1'(0.5) \approx -0.22275 < 0$, so since $\beta - \delta \approx 0.64838$, we have that for all $\beta - \delta \le x \le \beta$, $\psi_1'(x) < 0$. Consequently, function ψ_1 is decreasing and since $\psi_1(\beta) = 0$ the claim follows.

Now we consider the last point $(1, 2\beta)$ $(d = 5)$. Obviously, the case $y > (1/2 + x^\alpha)^{1/\alpha}$ prevails and by Lemma A.1, it suffices to show that $\psi_2(x) = \phi(x, (1/2 + x^\alpha)^{1/\alpha}) \ge 0$, for appropriate x's. By the argument above the derivative $\psi_2'(x)$ is a decreasing function of x. In our case $1 - \delta \le x \le 1$ and we can easily show that $\psi_2(1) = \phi(1, 2\beta) = 0$. It suffices to show that $\psi_2'(x) < 0$ when $1-\delta \le x \le 1$. Using the formula for ψ_2' above, we can show that $\psi_2'(0.8) \approx -0.012787 < 0$, so $\psi_2'(x) < 0$ for all $x > 0.8$ and we since have $x \ge 1 - \delta = 0.997$, we are done.

A.3 Unsuccessful areas in case (c) In this case our program does not succeed in a close area around points $(0, 1)$ $(d = 4, 5)$ and $(1, 2\beta)$ (for $d = 5$). A safe margin around these points can be set to $\delta = 0.0005$. Thus we have to prove that $\phi(x, y) \ge 0$ for:

- $0 \le x \le \delta$ and $1 \le y \le 1+\delta$ (in both cases $d = 4, 5$).
- $1 - \delta \le x \le 1$ and $2\beta - \delta \le y \le 2\beta + \delta$ $(d = 5)$.

One can again using exact formulas for α, β, γ easily show that $\phi(0, 1) = 0$ $(d = 4, 5)$, $\phi(1, 2\beta) = 0$ $(d = 5)$.

The derivative of function ϕ is

$$\frac{\partial \phi}{\partial y} = y^{\alpha-1}(y^\alpha - x^\alpha)^{(1/\alpha)-1} + y^{\alpha-1}(y^\alpha - 1)^{(1/\alpha)-1} - 1.$$

The first term in $\frac{\partial\phi}{\partial y}$ is $y^{\alpha-1}(y^\alpha - x^\alpha)^{(1/\alpha)-1} = (1 - x^\alpha/y^\alpha)^{(1/\alpha)-1}$. Since $y > 1$ (by (c)), also $y > x$, and so $x^\alpha/y^\alpha < 1$, which implies $1 - x^\alpha/y^\alpha < 1$, and $(1 - x^\alpha/y^\alpha)^{(1/\alpha)-1} > 1$. Also, the second term in $\frac{\partial\phi}{\partial y}$ is non-negative. This means the following.

Lemma A.2. *Under the assumptions in case (c), we have that* $\frac{\partial\phi}{\partial y} > 0$.

Since $y > (1 + x^\alpha/2)^{1/\alpha}$, by Lemma A.2 it suffices to show that $\psi(x) = \phi(x, (1 + x^\alpha/2)^{1/\alpha}) \ge 0$ for appropriate values of x. We have $\psi(x) = (1-x^\alpha/2)^{1/\alpha} - \psi(x) = (1-x^\alpha/2)^{1/\alpha} - (1+x^\alpha/2)^{1/\alpha} + \beta x$, and $\psi'(x) = -1/2\,(1/x^\alpha - 1/2)^{(1/\alpha)-1} - 1/2\,(1/x^\alpha + 1/2)^{(1/\alpha)-1} + \beta$. By $1/\alpha - 1 < 0$ the two first terms in $\psi'(x)$ are decreasing functions of x. Thus, we have that $\psi'(x)$ is also a decreasing function.

Consider first the point $(0, 1)$ in both cases $d = 4, 5$. We find now a large enough x such that $\psi'(x) \ge 0$. We have that $\psi'(x) \ge 0$ is equivalent to

$$(1/x^\alpha - 1/2)^{(1/\alpha)-1} + (1/x^\alpha + 1/2)^{(1/\alpha)-1} \le 2\beta \quad \Leftarrow$$

$$2(1/x^\alpha - 1/2)^{(1/\alpha)-1} \le 2\beta \quad \Leftrightarrow$$

$$1/\beta \le (1/x^\alpha - 1/2)^{(\alpha-1)/\alpha} \quad \Leftrightarrow$$

$$(1/(\beta^\alpha))^{1/(\alpha-1)} \le 1/x^\alpha - 1/2 \quad \Leftrightarrow$$

$$2^{1/(\alpha-1)} + 1/2 \le 1/x^\alpha \quad \Leftrightarrow \quad x \le \left(\frac{2}{2^{\alpha/(\alpha-1)} + 1}\right)^{1/\alpha}.$$

It can be checked that $\left(\frac{2}{2^{\alpha/(\alpha-1)}+1}\right)^{1/\alpha} \approx 0.454822$ when $d = 4$, and also when $d = 5$ we have $\left(\frac{2}{2^{\alpha/(\alpha-1)}+1}\right)^{1/\alpha} \approx 0.432845$. Since ψ' is a decreasing function, we have that for all $x \le 0.43$, $\psi'(x) \ge 0$. Since $\psi(0) = 0$, and $0 \le x \le \delta = 0.0005$, the claim follows.

Finally, consider the point $(1, 2\beta)$ in case of $d = 5$. In this case we want to show that $\psi(x) \ge 0$ when $1 - \delta \le x \le 1$. Since $\psi(1) = 0$, it suffices to prove that $\psi'(x) \le 0$ within this range of x. To do so, we will find now a small enough x such that $\psi'(x) \le 0$. We have that $\psi'(x) \le 0$ is equivalent to

$$(1/x^\alpha - 1/2)^{(1/\alpha)-1} + (1/x^\alpha + 1/2)^{(1/\alpha)-1} \ge 2\beta \quad \Leftarrow$$

$$2(1/x^\alpha + 1/2)^{(1/\alpha)-1} \ge 2\beta \quad \Leftrightarrow$$

$$1/\beta \ge (1/x^\alpha + 1/2)^{(\alpha-1)/\alpha} \quad \Leftrightarrow$$

$$(1/(\beta^\alpha))^{1/(\alpha-1)} \ge 1/x^\alpha + 1/2 \quad \Leftrightarrow$$

$$2^{1/(\alpha-1)} - 1/2 \ge 1/x^\alpha \quad \Leftrightarrow \quad x \ge \left(\frac{2}{2^{\alpha/(\alpha-1)} - 1}\right)^{1/\alpha}.$$

Now, we have $\left(\frac{2}{2^{\alpha/(\alpha-1)}-1}\right)^{1/\alpha} \approx 0.525747$ when $d = 5$. Since ψ' is a decreasing function, for all $x \ge 0.5258$, $\psi'(x) \le 0$. Since $\psi(1) = 0$, and $0.9995 = 1 - \delta \le x \le 1$, the claim follows.

Case of $d = 4$. In this case the tight ratio in characterized through Lemmas 2.1 and 2.2. If we make the two expressions there equal, then this gives $\beta = \frac{\sqrt{5d^2-8d+4}+2-d}{2d} \approx 0.651388$. Then since $\beta = (1/2)^{1/\alpha}$, $\alpha = \log(0.5)/\log(\beta) \approx 1.617047$, and by (7.1) $\gamma = \frac{d\beta-1}{d-1} \approx 0.535184$.

Case of $d = 5$. Here we use Lemma 2.4. We have

$$\beta = (1/2)^{1/\alpha} = (3/2)^{1/\alpha}/2 \quad \text{and} \quad \alpha = log_2 3,$$

which gives $\alpha \approx 1.584963$, $\beta \approx 0.64576$, and $\gamma = \frac{d\beta-1}{d-1} \approx 0.5572$. This finally proves Theorem 2.1.