Algorithmic Game Theory, Summer 2017 — Lecture 2 (5 pages)

Mixed Nash Equilibria

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In this lecture, we introduce the general framework of games. Congestion games, as introduced in the last lecture, are a special case. The notion of pure Nash equilibria readily generalizes but pure Nash equilibria might not exist. Therefore, we will introduce the concept of mixed Nash equilibria, which always exist in games with finitely many players and finitely many strategies.

1 Normal Form Game

Definition 2.1. *A* (normal form, payoff maximization) game *is a triple* $(\mathcal{N}, (S_i)_{i \in N}, (u_i)_{i \in N})$ *where*

- $\mathcal{N}$ *is the set of players,* $n = |\mathcal{N}|$,
- S_i *is the set of* (pure) strategies *of player* i,
- $S = \prod_{i \in \mathcal{N}} S_i$ *is the set of states,*
- $u_i \colon S \to \mathbb{R}$ *is the payoff/utility function of player* $i \in \mathcal{N}$*. In state* $s \in S$*, player* i *receives a payoff of* $u_i(s)$.

We denote by $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$ a state s without the strategy s_i. This notation allows us to concisely define a unilateral deviation of a player. For $i \in \mathcal{N}$, let $s \in S$ and $s'_i \in S_i$, then $(s'_i, s_{-i}) = (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)$.

Games with two players with finitely many strategies can be described by two matrices $A = (a_{s_1,s_2})_{s_1 \in S_1, s_2 \in S_2}$ and $B = (b_{s_1,s_2})_{s_1 \in S_1, s_2 \in S_2}$ (*bimatrix game*). Player 1 (referred to as row player) chooses a row; player two (column player) chooses a column. Their payoffs are given as $u_1(s) = a_{s_1,s_2}$, $u_2(s) = b_{s_1,s_2}$.

Example 2.2 (Battle of the Sexes)**.** *Suppose Angelina and Brad go to the movies. Angelina prefers watching movie A, Brad prefers watching movie B. However, both prefer watching a movie together to watching movies separately.*

	A	*B*
A	4 5	1 1
B	0 0	5 4

2 Pure Nash Equilibrium

Definition 2.3. *A strategy* s_i *is called a* best response *for player* $i \in \mathcal{N}$ *against a collection of strategies* s_{-i} *if* $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ *for all* $s'_i \in S_i$.

Note: A strategy s_i is a dominant strategy if and only if s_i is a best response for all s_{-i}.

Definition 2.4. *A state $s \in S$ is called a* pure Nash equilibrium *if s_i is a best response against the other strategies s_{-i} for every player $i \in \mathcal{N}$.*

So, a pure Nash equilibrium is stable against unilateral deviation. No player can reduce his cost by only changing his own strategy.

Pure Nash equilibria need not be unique.

Example 2.5 (Battle of the Sexes)**.** *We can find its pure Nash equilibria (A, B) and (B, A) by marking best responses with boxes.*

	A	B
A	$\boxed{2}$, $\boxed{1}$	6, 6
B	5, 5	$\boxed{1}$, $\boxed{2}$

A state is a Nash equilibrium if and only if it is marked for every player.

Not every game has a pure Nash equilibrium.

Example 2.6 (Rock-Paper-Scissors)**.** *The well-known game rock-paper-scissors can be represented by the following payoff matrix.*

	R	P	S
R	0, 0	-1, $\boxed{1}$	$\boxed{1}$, -1
P	$\boxed{1}$, -1	0, 0	-1, $\boxed{1}$
S	-1, $\boxed{1}$	$\boxed{1}$, -1	0, 0

There is no pure Nash equilibrium: In each of the nine states, at least one of the two players does not play a best response.

3 Mixed Nash Equilibrium

Definition 2.7. *A* mixed strategy *σ_i for player i is a probability distribution over the set of pure strategies S_i.*

We will only consider the case of finitely many pure strategies and finitely many players. In this case, we can write a mixed strategy σ_i as $(\sigma_{i,s_i})_{s_i \in S_i}$ with $\sum_{s_i \in S_i} \sigma_{i,s_i} = 1$. The payoff of a mixed state σ for player i is

$$u_i(\sigma) = \sum_{s \in S} p(s) \cdot u_i(s) \ ,$$

where $p(s) = \prod_{i \in \mathcal{N}} \sigma_{i,s_i}$ is the probability that the outcome is pure state s.

Definition 2.8. *A mixed strategy* σ_i *is a* (mixed) best-response strategy against a collection of mixed strategies σ_{-i} *if* $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})$ *for all other mixed strategies* σ'_i*.*

Definition 2.9. *A mixed state* σ *is called a* mixed Nash equilibrium *if* σ_i *is a best-response strategy against* σ_{-i} *for every player* $i \in \mathcal{N}$*.*

Note that every pure strategy is also a mixed strategy and every pure Nash equilibrium is also a mixed Nash equilibrium.

It is enough to only consider deviations to pure strategies.

Lemma 2.10. *A mixed strategy* σ_i *is a best-response strategy against* σ_{-i} *if and only if* $u_i(\sigma_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i})$ *for all pure strategies* $s'_i \in S_i$*.*

Proof. The "only if" part is trivial: Every pure strategy is also a mixed strategy.

For the "if" part, let σ_{-i} be an arbitrary mixed strategy profile for all players except for i. Furthermore, let σ_i be a mixed strategy for player i such that $u_i(\sigma_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i})$ for all pure strategies $s'_i \in S_i$.

Observe that for any mixed strategy σ'_i, we have $u_i(\sigma'_i, \sigma_{-i}) = \sum_{s'_i \in S_i} \sigma'_{i,s'_i} u_i(s'_i, \sigma_{-i}) \leq \max_{s'_i \in S_i} u_i(s'_i, \sigma_{-i})$. Using that $u_i(s'_i, \sigma_{-i}) \leq u_i(\sigma_i, \sigma_{-i})$ for all $s'_i \in S_i$, we are done. $\square$

While pure Nash equilibria do not necessarily exist, mixed Nash equilibria always exist if the number of players and the number of strategies is finite.

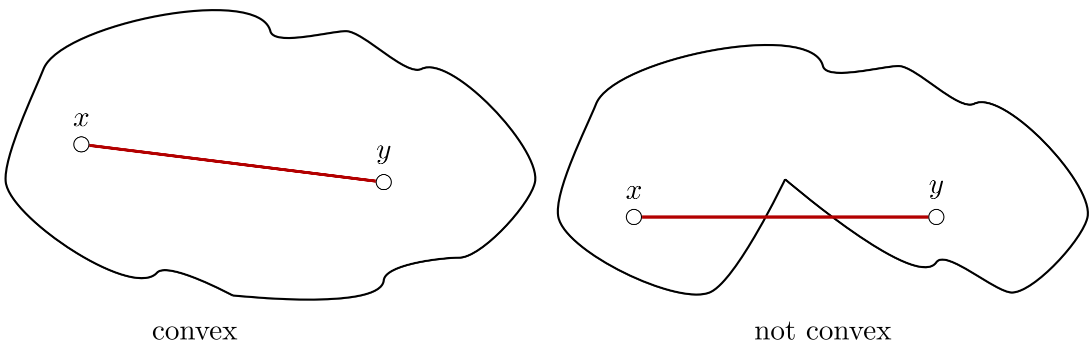
Theorem 2.11 (Nash's Theorem)**.** *Every finite normal form game has a mixed Nash equilibrium.*

Nash's theorem is usually proved via Brouwer's fixed point theorem.

Theorem 2.12 (Brouwer's Fixed Point Theorem)**.** *Every continuous function* $f\colon D \to D$ *mapping a compact and convex nonempty subset* $D \subseteq \mathbb{R}^m$ *to itself has a fixed point* $x^* \in D$ *with* $f(x^*) = x^*$*.*

As a reminder, these are the definitions of the terms used in Brouwer's fixed point theorem. Here, $\|\cdot\|$ denotes an arbitrary norm, for example, $\|x\| = \max_i |x_i|$.

- A set $D \subseteq \mathbb{R}^m$ is *convex* if for any $x, y \in D$ and any $\lambda \in [0, 1]$ we have $\lambda x + (1 - \lambda) y \in D$.



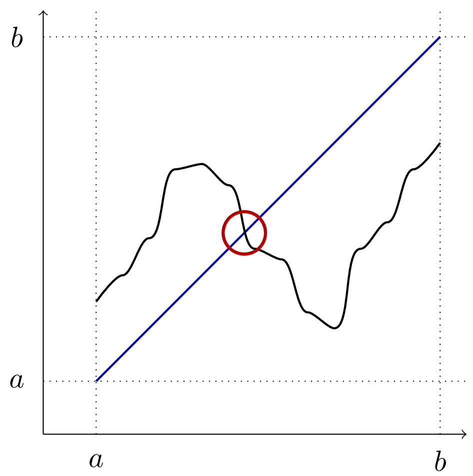
- A set $D \subseteq \mathbb{R}^m$ is *compact* if and only if it is closed and bounded.
- A set $D \subseteq \mathbb{R}^m$ is *bounded* if and only if there is some bound $r \geq 0$ such that $\|x\| \leq r$ for all $x \in D$.

- A set $D \subseteq \mathbb{R}^m$ is *closed* if it contains all its limit points. That is, consider any convergent sequence $(x_n)_{n \in \mathbb{N}}$ within D, i.e., $\lim_{n\to\infty} x_n$ exists and $x_n \in D$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n \in D$.

$$[0,1] \text{ is closed and bounded}$$
$$(0,1] \text{ is not closed but bounded}$$
$$[0,\infty) \text{ is closed and unbounded}$$

- A function $f\colon D \to \mathbb{R}^m$ is *continuous at a point* $x \in D$ if for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $y \in D$: If $\|x - y\| < \delta$ then $\|f(x) - f(y)\| < \epsilon$.
  f is called *continuous* if it is continuous at every point $x \in D$.

Equivalent formulation of Brouwer's fixed point theorem in one dimension:
For all $a, b \in \mathbb{R}$, $a < b$, every continuous function $f\colon [a,b] \to [a,b]$ has a fixed point.



4 Bonus: Proof of Nash Theorem

Proof of Theorem 2.11. Consider a finite normal form game. Without loss of generality let $\mathcal{N} = \{1,\ldots,n\}$, $S_i = \{1,\ldots,m_i\}$. So the set of mixed states X can be considered a subset of $\mathbb{R}^m$ with $m = \sum_{i=1}^n m_i$.

Exercise: Show that X is convex and compact.

We will define a function $f\colon X \to X$ that transforms a mixed strategy profile into another mixed strategy profile. The fixed points of f are shown to be the mixed Nash equilibria of the game.

For mixed state x and for $i \in \mathcal{N}$ and $j \in S_i$, let

$$\phi_{i,j}(x) = \max\{0, u_i(j, x_{-i}) - u_i(x)\} \ .$$

So, $\phi_{i,j}(x)$ is the amount by which player i's payoff would increase when unilaterally moving from x to j if this quantity is positive, otherwise it is 0.

Observe that by Lemma 2.10 a mixed state x is a Nash equilibrium if and only if $\phi_{i,j}(x) = 0$ for all $i = 1,\ldots,n$, $j = 1,\ldots,m_i$.

Define $f\colon X \to X$ with $f(x) = x' = (x'_{1,1}, ..., x'_{n,m_n})$ by

$$x'_{i,j} = \frac{x_{i,j} + \phi_{i,j}(x)}{1 + \sum_{k=1}^{m_i} \phi_{i,k}(x)}$$

for all $i = 1, \dots, n$ and $j = 1, \dots, m_i$.

Observe that $x' \in X$. That means, $f\colon X \to X$ is well defined. Furthermore, f is continuous. Therefore, by Theorem 2.12, f has a fixed point, i.e., there is a point $x^* \in X$ such that $f(x^*) = x^*$.

We only need to show that every fixed point x^* of f is a mixed Nash equilibrium. So, in other words, we need to show that $f(x^*) = x^*$ implies that $\phi_{i,j}(x^*) = 0$ for all $i = 1, \dots, n$, $j = 1, \dots, m_i$.

Fix some $i \in \mathcal{N}$. Once we have shown that $\phi_{i,j}(x^*) = 0$ for $j = 1, \dots, m_i$, we are done.

Let j' be chosen such that $u_i(j', x^*_{-i})$ is minimized among the j' such that $x^*_{i,j'} > 0$. As $u_i(x^*)$ is defined to be $\sum_{j=1}^{m_i} x^*_{i,j} \cdot u_i(j, x^*_{-i})$, we have $u_i(x^*) = \sum_{j=1}^{m_i} x^*_{i,j} \cdot u_i(j, x^*_{-i}) \geq \sum_{j=1}^{m_i} x^*_{i,j} \cdot u_i(j', x^*_{-i}) = u_i(j', x^*_{-i})$. Therefore $\phi_{i,j'}(x^*) = \max\{0, u_i(j', x^*_{-i}) - u_i(x^*)\} = 0$.

We now use the fact that x^* is a fixed point. Therefore, we have

$$x^*_{i,j'} = \frac{x^*_{i,j'} + \phi_{i,j'}(x^*)}{1 + \sum_{k=1}^{m_i} \phi_{i,k}(x^*)} = \frac{x^*_{i,j'}}{1 + \sum_{k=1}^{m_i} \phi_{i,k}(x^*)} \quad .$$

As $x^*_{i,j'} > 0$, we also have

$$1 = \frac{1}{1 + \sum_{k=1}^{m_i} \phi_{i,k}(x^*)} \quad ,$$

and so

$$\sum_{k=1}^{m_i} \phi_{i,k}(x^*) = 0 \quad .$$

Since $\phi_{i,k}(x^*) \geq 0$ for all k, we have to have $\phi_{i,k}(x^*) = 0$ for all k. This completes the proof. $\square$

Recommended Literature

- Philip D. Straffin. Game Theory and Strategy, The Mathematical Association of America, fifth printing, 2004. (For basic concepts)
- J. Nash. Non-Cooperative Games. The Annals of Mathematics 54(2):286-295. (Nash's original paper)