Algorithmic Game Theory, Summer 2017 — Lecture 1 (5 pages)

Introduction to Congestion Games

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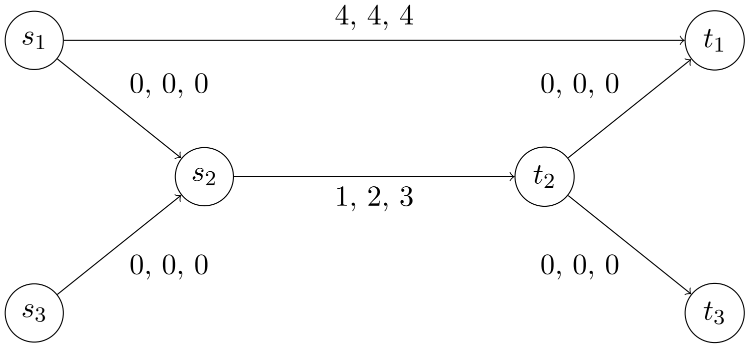
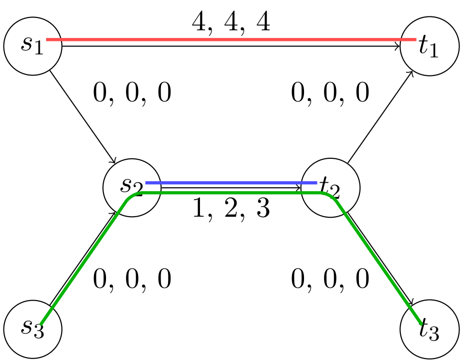
In this lecture, we get to know *congestion games*, which will be our running example for many concepts in game theory. Before coming to the formal definition, let us consider the following example.

We are given the following directed graph; there are three players, who each want to reach their respective destination node from their start node. Edge labels indicate the cost *each* player incurs if this edge is used by one, two, or all three players. So, if the edge label is a, b, c and the edge is used by two players, then each player has cost b for this edge.



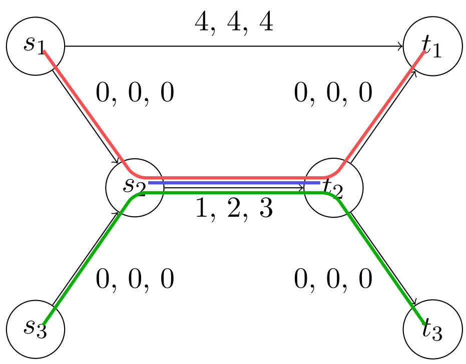
Players 2 and 3 do not have any choice, but player 1 has. He can either use the direct edge or go via s_2 and t_2. That is, we have the following two states.

State A:



social cost: $4 + 2 + 2 = 8$

State B:



social cost: $3 + 3 + 3 = 9$

We observe that State A has a smaller *social cost* than State B. However, player 1 prefers State B because his *individual cost* is smaller there. In contrast to State A, State B is stable; it is an equilibrium.

We will introduce a general model that allows us to capture these effects. We will ask questions such as: Are there equilibria? How can these equilibria be found? How much performance is lost due to selfishness?

1 Formal Definition

Definition 1.1 (Congestion Game (Rosenthal 1973))**.** *A* congestion game *is a tuple* $\Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_r)_{r \in \mathcal{R}})$ *with*

- $\mathcal{N} = \{1, \dots, n\}$*, set of players*

- $\mathcal{R} = \{1, \dots, m\}$, *set of resources*
- $\Sigma_i \subseteq 2^{\mathcal{R}}$, *strategy space of player* i
- $d_r \colon \{1, \dots, n\} \to \mathbb{Z}$, *delay function or resource* r

For any state $S = (S_1, \dots, S_n) \in \Sigma_1 \times \cdots \Sigma_n$,

- $n_r(S) = |\{i \in \mathcal{N} \mid r \in S_i\}|$*: number of players with* $r \in S_i$
- $d_r(n_r(S))$*: delay of resource* r
- $\delta_i(S) = \sum_{r \in S_i} d_r(n_r)$*: delay of player* i

The cost *of player* i *in state* S *is* $c_i(S) = \delta_i(S)$*, that is, players aim at minimizing their delays.*

Our above example is a *network* congestion game: There is a graph $G = (V, E)$ with dedicated source-sink pairs $(s_1, t_1), \dots, (s_n, t_n)$ such that Σ_i is the set of s-t paths.

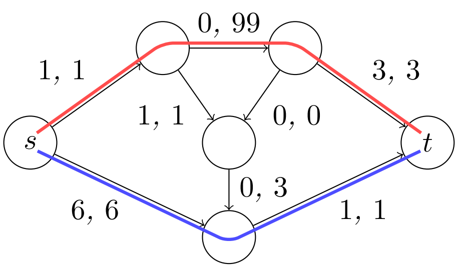
Definition 1.2. *A strategy* S_i *is called a* best response *for player* $i \in \mathcal{N}$ *against a profile of strategies* $S_{-i} := (S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n)$ *if* $c_i(S_i, S_{-i}) \leq c_i(S'_i, S_{-i})$ *for all* $S'_i \in \Sigma_i$*. A state* $S \in \Sigma_1 \times \cdots \times \Sigma_n$ *is called a* pure Nash equilibrium *if* S_i *is a best response against the other strategies* S_{-i} *for every player* $i \in \mathcal{N}$*.*

2 Existence of Pure Nash Equilibria

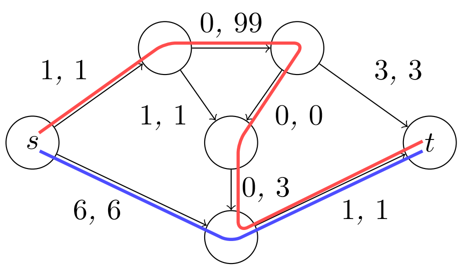
As our first result, we will show every congestion game has a pure Nash equilibrium. We will talk about *improvement steps*. The pair of states (S, S') is an improvement step if there is some player $i \in \mathcal{N}$ such that $c_i(S') < c_i(S)$ and $S'_{-i} = S_{-i}$.

Example 1.3. *A sequence of (best response) improvement steps:*

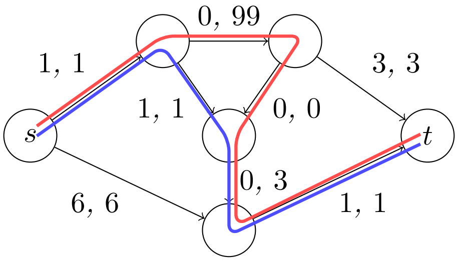
start:



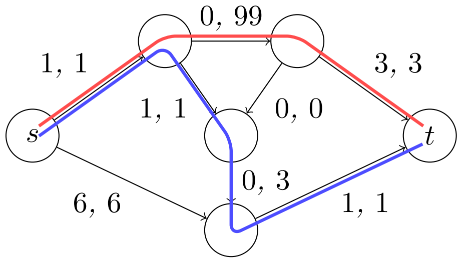
after first improvement (red player):



after second improvement (blue player):



after third improvement (red player):



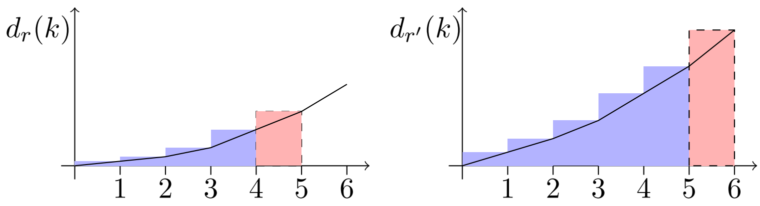


Figure 1: Proof of Lemma 1.6: The contribution of two resources r and r' to the potential is the shaded area. If a player changes from r' to r, his delay changes exactly as the potential value (difference of red areas).

Theorem 1.4 (Rosenthal 1973)**.** *For every congestion game, every sequence of improvement steps is finite.*

This result immediately implies

Corollary 1.5. *Every congestion game has at least one pure Nash equilibrium.*

Proof of Theorem 1.4. Rosenthal's analysis is based on a potential function argument. For every state S, let

$$\Phi(S) = \sum_{r \in \mathcal{R}} \sum_{k=1}^{n_r(S)} d_r(k) \enspace .$$

This function is called *Rosenthal's potential function.*

Lemma 1.6. *Let S be any state. Suppose we go from S to a state S' by an improvement step of player i. then $\Phi(S') - \Phi(S) = c_i(S') - c_i(S)$.*

Proof. The potential $\Phi(S)$ can be calculated by inserting the players one after the other in any order, and summing the delays of the players at the point of time at their insertion.

Without loss of generality player i is the last player that we insert when calculating $\Phi(S)$. Then the potential accounted for player i corresponds to the delay of player i in state S. When going from S to S', the delay of i decreases by Δ, and, hence, Φ decreases by Δ as well (see Figure 2 for an example. □

The lemma shows that Φ is a so-called *exact potential*, i.e., if a single player decreases its latency by a value of $\Delta > 0$, then Φ decreases by exactly the same amount.

Further observe that

(i) the delay values are integers so that, for every improvement step, $c_i(S') - c_i(S) \leq -1$,

(ii) for every state S, $\Phi(S) \leq \sum_{r \in \mathcal{R}} \sum_{i=1}^{n} |d_r(i)|$,

(iii) for every state S, $\Phi(S) \geq -\sum_{r \in \mathcal{R}} \sum_{i=1}^{n} |d_r(i)|$.

Consequently, the number of improvements is upper-bounded by $2 \cdot \sum_{r \in \mathcal{R}} \sum_{i=1}^{n} |d_r(i)|$ and hence finite. □

3 Convergence Time of Improvement Steps

Rosenthal's theorem shows that any sequence of improvement steps is finite. However, it does not give any guarantee how many improvement steps are needed to reach a Nash equilibrium.

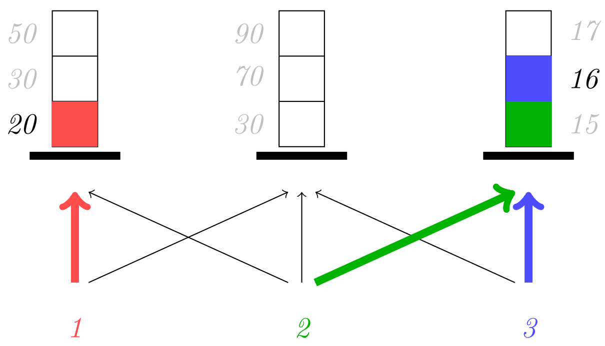
A trivial upper bound on the length of any (finite) sequence of improvement steps is the overall number of states, which is at most 2^{mn}. However, this is only a very poor guarantee and by no means tight.

We will show a significantly better, namely polynomial, guarantee for *singleton congestion games*. In this subclass of congestion games every player wants to allocate only a single resource at a time from a subset of allowed resources. Formally:

Definition 1.7 (Singleton Games)**.** *A congestion game is called* singleton *if, for every $i \in \mathcal{N}$ and every $R \in \Sigma_i$, it holds that $|R| = 1$.*

Although this constraint on the strategy sets is quite restrictive, there are still up to m^n different states.

Example 1.8 (Singleton Congestion Game)**.** *Consider a "server farm" with three servers a, b, c (resources) and three players 1,2,3 each of which wants to access a single server.*



The colored arrows indicate a pure Nash equilibrium.

Theorem 1.9. *In a singleton congestion game with n players and m resources, all improvement sequences have length $O(n^2m)$.*

Proof idea:

- Replace original delays by bounded integer values without changing the preferences of the players.
- Upper bound on the maximum potential wrt new delays.
- Due to integer values, decrease of potential in an improvement step is at least 1. Hence, length of every improvement sequence is bounded by maximum potential.

Proof. Sort the set of delay values $V = \{d_r(k) \mid r \in \mathcal{R}, 1 \leq k \leq n\}$ in increasing order. Define alternative, new delay functions:

$$\bar{d}_r(k) := \text{position of } d_r(k) \text{ in sorted list.}$$

The new delay of a player i using resource r in state S is $\bar{\delta}_i(S) = \bar{d}_r(n_r(S))$.

Observation 1.10. *Let S and S' be two states such that (S, S') is an improvement step for some player i with respect to the original delays. Then (S, S') is an improvement step for i with respect to the new delays, as well.*

Furthermore, observe that $\bar{d}_r(k) \leq nm$ for all $r \in \mathcal{R}$ and $k \in [n]$ because there are at most nm elements in V. Therefore, Rosenthal's potential function with respect to the new delays $\bar{d}_r(k)$ can be upper-bounded as follows:

$$\bar{\Phi}(S) = \sum_{r\in\mathcal{R}} \sum_{k=1}^{n_r(S)} \bar{d}_r(k) \leq \sum_{r\in\mathcal{R}} \sum_{k=1}^{n_r(S)} n\,m \leq n^2\,m \enspace ,$$

where in the last step we use $\sum_{r\in\mathcal{R}} n_r(S) = n$ because every player uses exactly one resource.

It holds that $\bar{\Phi} \geq 1$. Also, $\bar{\Phi}$ decreases by at least 1 in every step. Therefore, the length of every improvement sequence is upper-bounded by n^2m. □

Example 1.11. *The sorted list of delay values in Example 1.8 is*

$$15, 16, 17, 20, 30, 50, 70, 90.$$

Hence, the old and new delay functions are

$$\begin{aligned} d_a(1,2,3) &= (20,30,50) & \bar{d}_a(1,2,3) &= (4,5,6)\\ d_b(1,2,3) &= (30,70,90) & \bar{d}_b(1,2,3) &= (5,7,8)\\ d_c(1,2,3) &= (15,16,17) & \bar{d}_c(1,2,3) &= (1,2,3) \end{aligned}$$

Recommended Literature

- D. Monderer, L. Shapley. Potential Games. Games and Economic Behavior, 14:1124–1143, 1996. (Equivalence Congestion and Potential Games)
- H. Ackermann, H. Röglin, B. Vöcking. On the impact of combinatorial structure on congestion games. Journal of the ACM, 55(6), 2008. (Generalization of Theorem on Singleton Games)